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SELECT
EXERCISES
FOR
YOUNG PROFICIENTS
IN THE
MATHEMATICKS.

CONTAINING

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| I. A great variety of Algebraical Problems with their Solutions. | Gunnery, independent of the Conic Sections. |
| II. A choice Number of Geometrical Problems with their Solutions both Algebraical and Geometrical. | IV. A new and very comprehensive Method for finding the Roots of Equations in Numbers. |
| III. THE Theory of | V. A short Account of the Nature and first Principles of Fluxions. |

By THOMAS SIMPSON, F. R. S.

A NEW EDITION.

TO WHICH IS PREFIXED,
AN ACCOUNT of the LIFE and WRITINGS of the AUTHOR,

By CHARLES HUTTON, F. R. S.

LONDON:

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Successor to Mr. NOURSE, in the Strand.

MDCXCII.

MA THEMATICALLY
IN THE
YOUNG PROICIENTS
FOR
X E R C I S E
O C T I E R



TO
JOHN BACON,
Of *Newtoncap*, Esq; F. R. S.

S I R,

WHEN Gentlemen of your Station and Figure become the Patrons of Science it is a Benefit to the Publick, their Expectations of farther Improvements having then the best Foundation. And All who have the Pleasure of your Acquaintance, and know your Attachment to polite and useful Learning, in which a Knowledge of the Mathematicks may be justly included, will be sensible of my Happinefs in being thus permitted to address You.

Believe

Believe me, Sir, whatever may be
the Fate of these Sheets, I shall, at
all Times, consider this Use of your
Name as a singular Honour to,

SIR,

Your most obedient, and most

Humble Servant,

T. SIMPSON.

Woolwich, May 1,

1752.

THE
P R E F A C E.

THE ensuing Work, or at least the greatest part of it, was originally composed for my own use in the ROYAL ACADEMY: And it is upon a presumption that it may also be of service to others, especially those employed in a like public way of teaching, that it now appears in the World.

The Work itself consists of six * distinct parts, or tracts; each of which I shall here give some account of.

The first part contains a number of Algebraical Problems, with their Solutions; designed as proper exercises for young beginners. In the course of these Problems and Solutions (whereof the greater part will appear to be new) the art of managing equations, and the various methods of substitution are taught and illustrated.

The

* The sixth part, relating to the Doctrine of Annuities is omitted in this Edition, and added to the Author's Treatise on that subject,

The second part comprehends a variety of Geometrical Problems with their Solutions, both by Algebra and also independent of it, from principles purely Geometrical. In this part the learner will find a large field to exercise his industry in: he will moreover have the opportunity of comparing the two methods of solution together, and from thence observing, that sometimes the one has the advantage, and sometimes the other; that, in some cases they both proceed upon the very same properties, and in others, upon quite different ones: and it may be further remarked from hence (which will be of some use to know) that, when quantities are given in magnitude only, the Algebraic method generally claims the preference, in point of ease and expedition, at least; whereas the advantage is almost always on the side of the geometrical effect, when the positions of points and lines, and the quantities of angles are given.

There is, however, one particular, or two, in this part, that may be thought to stand in need of some apology.

In the first place, the frequent use of symbols, common to the Algebraic notation, may, perhaps, be looked upon as repugnant to the rigour and strictness of Geometry. But it is not the use of symbols (which some, more scrupulous

pulous than discerning, have condemned) but the ideas annexed to them, that render the consideration Geometrical, or Ungeometrical. In pure Geometry regard is always had to the absolute quantity of some one of the three kinds of extension, abstractedly considered; and, whatever symbols are used here, are to be considered as expressive of the quantities themselves, and not as any measures, or numerical values of them. Thus by $A \times B$, taken in a geometrical sense, we have an idea, not of the product of two numbers (as in the algebraic notation), but of a real, rectangular, space comprehended under two right-lines, represented by A and B, and two others equal to them. So, likewise, $\frac{B \times C}{A}$ is not to be understood here in the Light of an algebraic fraction, but as a right-line, which is fourth proportional to three other right-lines, represented by A, B and C.—These distinctions are absolutely necessary to those who would have an accurate idea of the subject.

The second particular, above hinted at, relates to the quotations; wherein I have referred to my own Elements of Geometry, and not to those of Euclid, so universally known and established. But for this there were two reasons: First, those persons, for whose instruction these sheets are, in a more particular manner,

manner, designed, are taught the first principles of Geometry from other Elements than those of Euclid; and, secondly, a number of propositions are used here that are only to be met with in modern authors.

In the third part the Theory of Gunnery, or the Motion of Projectiles, is considered, exclusive of the conic Sections; and the practical solutions of the several cases depending on the theory (as well those where the object is elevated or depressed as where it is situate in the plain of the horizon) are given, at large, by plane Trigonometry.

The fourth part exhibits a new, and very comprehensive method for extracting the roots of algebraical Equations; whereby the number sought may be determined, to any proposed degree of exactness, without the trouble of repeating the operation, as in the common way, by converging Series's.

The fifth part gives some account of the Nature of Fluxions, together with the investigation of the fundamental rules; and may be of use, not only to beginners, but also to such, who, though tolerably well versed in the practise and application of Fluxions, have nevertheless but an imperfect idea of the first principles of this difficult branch of science.

May, 1752.

MEMOIRS

MEMOIRS of the LIFE and WRITINGS
of the AUTHOR.

MR. THOMAS SIMPSON was born at *Market Bosworth*, in the county of *Leicester*, August the 20th, O. S. 1710. His father was a stuff-weaver in that town; and, though in tolerable circumstances, yet, intending to bring up his son Thomas to his own business, he took so little care of his education, that he was only taught to read English. But nature had furnished him with talents and a genius for very different pursuits, which led him afterwards to the highest rank in the mathematical and philosophical sciences.

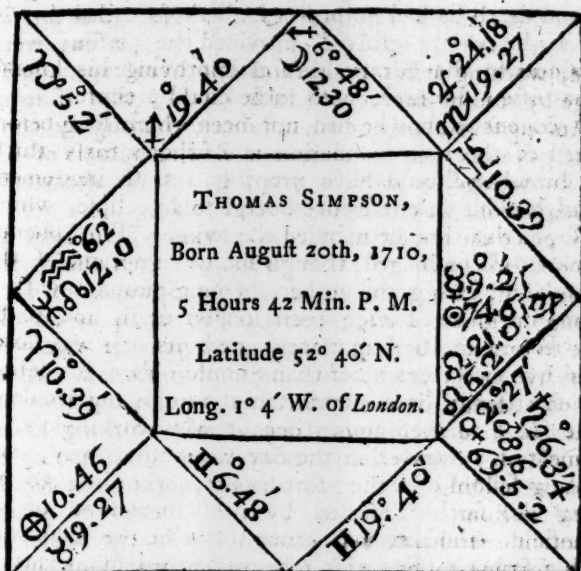
Young Simpson very soon gave indications of his turn for study in general, by eagerly reading all books he could meet with, teaching himself to write, and embracing every opportunity he could find of deriving knowledge from other persons. Thomas's father observing him thus to neglect his business, by spending his time in reading what he thought useless books, and following other such like pursuits, used all his endeavours to check such proceedings, and to induce him to follow his profession with steadiness and better effect. But after many struggles for this purpose, the differences thus produced between them at length rose to such a height, that our Author quitted his father's house entirely.

Upon this occasion he repaired to Nuneaton, a town at a small distance from Bosworth, where he went to lodge at the house of a taylor's widow, of the name of Swinfield, who had been left with two children, a daughter and a son, by her husband, of whom the son, who was the younger, being but about two years older than Simpson, had become his intimate friend and companion. And here he continued some

time, working at his trade, and improving his knowledge by reading such books as he could procure.

Among several other circumstances which, long before this, gave occasion to shew our Author's early thirst for knowledge, as well as proving a fresh incitement to acquire it, was that of a large solar eclipse, which took place on the 11th of May, 1724. This phenomenon, so awful to many who are ignorant of the cause of it, struck the mind of young Simpson with a strong curiosity to discover the reason of it, and to be able to predict the like surprising events. It was however five or six years before he could obtain his desire, which at length was gratified by the following accident. After he had been some time at Mrs. Swinfield's, at Nuneaton, a travelling pedlar came that way, and took a lodging at the same house, according to his usual custom. This man, to his profession of an itinerant merchant, had joined the more profitable one of a fortune-teller, which he performed by means of judicial astrology. Every one knows with what regard persons of such a cast are treated by the inhabitants of country villages; it cannot be surprising therefore that an untutored lad of nineteen should look upon this man as a prodigy, and, regarding him in this light, should endeavour to ingratiate himself into his favour; in which he succeeded so well, that the sage was no less taken with the quick natural parts and genius of his new acquaintance. The pedlar, intending a journey to Bristol fair, left in the hands of young Simpson an old edition of Cocker's Arithmetic, to which was subjoined a short Appendix on Algebra, and a Book upon Genitures, by Partridge the almanac maker. These books he had perused to so good purpose, during the absence of his friend, as to excite his amazement upon his return; in consequence of which he set himself about erecting the following genethliacal type, in order to a presage of Thomas's future fortune.

THOMAS



This position of the heavens having been maturely considered *secundum artem*, the wizard, with great confidence, pronounced, that "within two years time Simpson would turn out a greater man than himself."

In fact, our Author profited so well by the encouragement and assistance of the pedlar, afforded him from time to time when he occasionally came to Nuneaton, that, by the advice of his friend, he at length made an open profession of casting nativities himself; from which, together with teaching an evening school, he derived a pretty pittance, so that he greatly neglected his weaving, to which indeed he had never manifested any very great attachment, and soon became the oracle of Nuneaton, Bosworth, and the environs. Scarce a courtship advanced to a match, or a bargain to a sale, without previously consulting the infallible Simpson about the consequences. But helping folks to stolen goods, he always declared above his skill; and that as to

life and death he had no power : all those called *lawful questions* he readily resolved, provided the persons were certain as to the horary *data* of the horoscope : and, he has often declared, with such success, that if from very cogent reasons he had not been thoroughly convinced of the vain foundation and fallaciousness of his art, he never should have dropt it, as he afterwards found himself in conscience bound to do.

About this time he married the widow Swinfield, in whose house he lodged, though she was then almost old enough to be his grandmother, being upwards of fifty years of age, and having been looked upon as an old maid before her first marriage, and yet her youngest child was two years older than Simpson himself. After this the family lived comfortably enough together for some short time, Simpson occasionally working at his business of a weaver in the day-time, and teaching an evening school or telling fortunes at night ; the family being also farther assisted by the labours of young Swinfield, who had been brought up in the profession of his father. But this tranquillity was soon interrupted, and our Author driven at once from his home and the profession of astrology, by the following accident. A young woman in the neighbourhood had long wished to hear or know something of her lover, who had been gone to sea ; but Simpson had put her off from time to time, till the girl grew at last so importunate, that he could deny her no longer. He asked her if she would be afraid if he should raise the devil, thinking to deter her ; but she declared she feared neither ghost nor devil, so he was obliged to comply. The scene of action pitched upon was a barn, and young Swinfield was to act the devil or ghost, who being concealed under some straw in a corner of the barn, was, at a signal given, to rise slowly out from among the straw, with his face marked so that the girl might not know him. Every thing being in order, the girl came at the time appointed ; when Simpson, after cautioning her not to be afraid, began

began muttering some mystical words, and chalking round about them, till, on the signal given, up rises the taylor slow and solemn, to the great terror of the poor girl, who, before she had seen half his shoulders, fell into violent fits, crying out it was the very image of her lover; and the effect upon her was so dreadful, that it was thought either death or madness must be the consequence. So that poor Simpson was obliged immediately to abandon at once both his home and the profession of a conjuror.

Upon this occasion it would seem he fled to Derby, where he remained some two or three years, instructing pupils in an evening school, and working at his trade by day. Here too he wrote a song on occasion of the parliamentary election at this place, in the year 1733, in favour of the Cavendish family; and it is probable he continued here till about the year 1735 or 1736, as I have been informed by the late Mr. Whitehurst, who then knew our Author, having also come to settle at Derby about the same time with him.

It would seem indeed that Simpson had an early turn for versifying, both from the circumstance of the song abovementioned, and from his first two mathematical questions that were published in the Ladies Diary, which were both in a set of verses, not ill written for the occasion. These were printed in the Diary for 1736, and therefore must at latest have been written in the year 1735. These two questions, being at that time pretty difficult ones, shew the great progress he had even then made in the mathematics; and from an expression in the first of them, viz. where he mentions his residence as being in latitude 52° , it appears he was not then come up to London, though he must have done so very soon after.

Together with his astrology he had soon furnished himself with enough of arithmetic, algebra, and geometry to be qualified for looking into the Ladies Diary (of which he had afterwards for several years the direction), by which he came to understand that there

there was a still higher branch of the mathematical knowledge than any he had yet been acquainted with; and this was the method of *Fluxions*. But our young analyst was quite at a loss to discover any English author who had written on the subject, except Mr. Hayes; and his work being a folio, and then pretty scarce, exceeded his ability of purchasing: however, an acquaintance lent him Mr. Stone's *Fluxions*, which is a translation of the *Marquis de l'Hospital's Analyse des Infiniment Petits*: by this one book, and his own penetrating talents, he was, as we shall presently see, enabled in a very few years to compose a much more accurate treatise on this subject than any that had before appeared in our language.

After he had quitted astrology and its emoluments, he was driven to hardships for the subsistence of his family, while at Derby, notwithstanding his other industrious endeavours in his own trade by day, and teaching pupils at evenings. This determined him to repair to London, which he did in 1735 or 1736, as before said, leaving the family behind, his wife being then pregnant with her first child by our Author, which was contrary to the expectation of every one, being then about fifty-five years of age, for which reason her neighbours used to say, "there goes Sarah the wife of Abraham."

On his first coming to London, Mr. Simpson wrought for some time at his business in Spitalfields, and taught mathematics at evenings, or any spare hours. His industry turned to so good account, that he returned down into the country, and brought up his wife and three children, she having produced her first child in his absence. The number of his scholars increasing, and his abilities becoming in some measure known to the public, he was encouraged to make proposals for publishing by subscription, *A new Treatise of Fluxions: wherein the Direct and Inverse Method are demonstrated after a new, clear, and concise Manner, with their Application to Physics and Astronomy: also the Doctrine*
of

of Infinite Series and Reverting Series universally, are amply explained, Fluxionary and Exponential Equations solved: together with a Variety of new and curious Problems.

When Mr. Simpson first proposed his intentions of publishing such a work, he did not know of any English book, founded on the true principles of Fluxions, that contained any thing material, especially the practical part; and though there had been some very curious things done by several learned and ingenious gentlemen, the principles were nevertheless left obscure and defective, and all that had been done by any of them in *infinite series*, very inconsiderable.

The book was published in 4to, in the year 1737, although the author had been frequently interrupted from furnishing the press so fast as he could have wished, through his unavoidable attention to his pupils for his immediate support. The principles of fluxions treated of in this work, are demonstrated in a method accurately true and genuine, not essentially different from that of their great inventor, being entirely expounded by finite quantities. In the first and second parts are given a great many new, and some very curious examples in the solutions of problems, rendered plain to ordinary capacities.

The second part treats of *Infinite Series*. And here nothing is proposed without demonstration, and every thing is illustrated by easy examples. New rules are also laid down for finding the forms of series, without taking in any of the superfluous terms.

The third part contains a familiar method of finding and comparing fluents, illustrated with some useful and easy applications.

In the fourth part is shewn the use of fluxions in some of the most sublime branches of *Physics* and *Astronomy*; where, besides several things done in a method quite different from any thing to be met with in other authors, there are some very useful speculations

tations relating to the doctrine of Pendulums and Centripetal Forces.

To this is added a supplement; being a collection of miscellaneous problems, independant of the foregoing four parts; and containing, among other matters, an investigation of the Areas of Spherical Triangles; the Curve of Pursuit; the Paths of Shadows; the Motion of Projectiles in a Medium; and the manner of finding the Attractive Force of bodies of different forms, acting according to a given law.

In 1740, Mr. Simpson published a Treatise on *The Nature and Laws of Chance*, in 4to. To which is annexed, *Full and clear Investigations of two important Problems added in the 2d Edition of Mr. De Moivre's Book on Chances*, as also *two New Methods for the Summation of Series*.

Our Author's next publication was a 4to volume of *Essays on several curious and interesting Subjects in Speculative and Mixed Mathematics*; printed in the same year 1740: dedicated to FRANCIS BLAKE, Esq. since Fellow of the Royal Society, and our Author's good friend and patron.

The first of these Essays shews the Theory of the apparent Place of the Stars (commonly called their *Aberration*) arising from the progressive Motion of Light, and of the Earth in its Orbit; with practical Rules for computing the same, communicated by Dr. Bevis.

The 2d treats of the Motion of Bodies affected by projectile and centripetal Forces; in which the most considerable Matters in the first Book of Newton's *Principia* are clearly investigated.

The 3d is a Solution of Kepler's Problem, with a concise practical Rule.

The 4th treats of the Motion and Paths of Projectiles in resisting mediums; determining the most important particulars upon this head, in the 2d Book of the *Principia*.

The

The 5th considers the Resistances, Velocities, and Times of vibration of pendulous Bodies in Mediums.

The 6th contains a new Method of Solution of all Kinds of algebraical equations in numbers, more general than ever before given.

The 7th is concerning the Method of *Increments*, with examples.

The 8th is a concise Investigation of a Theorem for finding the Sum of a Series of Quantities, by Means of their Differences.

The 9th is a general Way of investigating the Sum of a recurring Series.

The 10th is a new and general Method for finding the Sum of any Series of Powers, whose Roots are in arithmetical Progression; being also applicable to Series of other Kinds.

The 11th relates to angular Sections, with some remarkable Properties of the circle.

The 12th shews an easy and expeditious Method of reducing a compound Fraction to simple ones.

The 13th, or last, which contains a general Quadrature of hyperbolical Curves, is a problem that had exercised the skill of several eminent Mathematicians. None of the solutions before published extended farther than to particular cases, except one, which is in the Philosophical Transactions, without demonstration, by Mr. Klingenshierna, professor of mathematics at Upsal. This is investigated by Mr. Simpson in two different ways; and the general construction is rendered remarkably easy, simple, and fit for practice. And, on this head it would seem that Mr. Klingenshierna was well pleased with what Mr. Simpson had done; for being afterwards appointed secretary to the Royal Academy at Stockholm, as a mark of his esteem, he procured a diploma, to be transmitted to him, by which he was constituted a member of that learned body.

Our Author's next work was, *The Doctrine of Annuities and Reversions, deduced from general and evident Principles:*

Principles: with useful Tables, shewing the Values of Single and Joint Lives, &c. in 8vo, 1742. This was followed, in 1743, by an *Appendix, containing some Remarks on a late Book on the same Subject* (by Mr. Abr. De Moivre, F. R. S.) with *Answers to some personal and malignant Representations in the Preface thereof.* To this answer Mr. De Moivre never thought fit to reply. A new edition of this work has lately been published, augmented with the tract upon the same subject that was printed in our Author's *Select Exercises*.

In 1743 also was published his *Mathematical Dissertations on a Variety of Physical and Analytical Subjects*, in 4to; containing, among other particulars,

A Demonstration of the true Figure which the Earth, or any Planet, must acquire from its rotation about an Axis. A general Investigation of the Attraction at the Surfaces of Bodies nearly spherical. A Determination of the Meridional Parts, and the Lengths of the several Degrees of the Meridian, according to the true Figure of the Earth. An Investigation of the Height of the Tides in the Ocean. A new Theory of Astronomical Refractions, with exact Tables deduced from the same. A new and very exact Method for approximating the Roots of Equations in Numbers; which quintuples the number of Places at each Operation. Several new Methods for the summation of Series. Some new and very useful Improvements in the Inverse Method of Fluxions. The work being dedicated to MARTIN FOLKES, Esq. president of the Royal Society.

His next book was *A Treatise of Algebra, wherein the fundamental Principles are demonstrated, and applied to the Solution of a Variety of Problems.* To which he added, *The Construction of a great Number of Geometrical Problems, with the Method of resolving them numerically.*

This work, which was designed for the use of young beginners, was inscribed to WILLIAM JONES, Esq. F. R. S. and printed in 8vo, 1745. And a new edition appeared in 1755, with additions and improvements;

ments; among which was a new and general Method of resolving all Biquadratic Equations, that are complete, or having all their terms. This edition was dedicatad to JAMES EARL of MORTON, F. R. S. Mr. JONES being then dead. The work has gone through several other editions since that time: the 6th, or last, was in 1790.*

His next work was, *Elements of Geometry, with their Application to the Mensuration of Superfices and Solids, to the determination of Maxima and Minima, and to the Construction of a great Variety of geometrical Problems*: first published in 1747, in 8vo. And a second edition of the same came out in 1760, with great alterations and additions, being in a manner a new work, designed for young beginners, particularly for the gentlemen educated at the Royal Military Academy at Woolwich, and dedicated to CHARLES FREDERICK, Esq. Surveyor General of the Ordnance. And other editions have appeared since.

Mr. Simpson met with some trouble and vexation in consequence of the first edition of his Geometry. First, from some reflections made upon it, as to the accuracy of certain parts of it, by Dr. Robert Simson, the learned professor of mathematicks in the University of Glasgow, in the notes subjoined to his edition of Euclid's Elements. This brought an answer to those remarks from Mr. Simpson, in the notes added to the 2d. edition as above; to some parts of which Dr. Simson again replied in his notes on the next edition of the said Elements of Euclid.

The second was by an illiberal charge of having stolen his Elements from Mr. Muller, the professor of fortification and artillery at the same academy at Woolwich, where our Author was professor of geometry and mathematics. This charge was made at the end of the preface to Mr. Muller's Elements of Mathematics, in two volumes, printed in 1748, in these words:

b 2

“ There

* A regular list of the Author's Publications will be given at the end.

" There has lately been published a book under the
 " title of Elements of Plane Geometry, designed for
 " the Use of Schools, which is an incorrect Copy of
 " the first eight Sections of this Work, lent the pre-
 " tended Author upon a particular Occasion, and
 " printed in a spurious manner, without my know-
 " ledge or consent; an Action too scandalous for
 " any Man of honour to be guilty of. The Editor
 " imagined, I suppose, that the changing some pro-
 " positions, and mangling the demonstrations of others,
 " was a sufficient Disguise to make it pass for his own
 " performance; but how far this will justify such a
 " piece of piracy, must be left to the judgement of
 " the public."

To this extraordinary charge, Mr. Simpson, in the
 preface to the 2d edition, replies, " Were I to attempt
 " to describe the ideas excited in my mind by the
 " singular modesty of this important and solemn
 " appeal to the publick, I should be at a loss for fit
 " words to express them without transgressing the
 " bounds of decency. But I hope that I have not
 " deserved so ill of the publick, to be thought capable
 " of acting so very humble a part, as that of copying
 " from *this* author, and of mangling *his* demonstra-
 " tions, in order to make them pass for *my own*.—
 " That a manuscript of his (containing between 20
 " and 30 of the principal Theorems in Geometry,
 " extremely ill digested) came into my hands, is in-
 " deed true; but it was not *lent me*, but forced upon
 " me, by himself (the very first night after my removal
 " to Woolwich) in virtue of an *article* in the original
 " rules and instructions for the Academy; whereby it
 " is ordered, that the second master shall teach geo-
 " metry under the direction of the first master. But
 " this well intended article, which has been made
 " subservient to the purposes of ignorant tyranny,
 " and daring calumny, has since, in consequence of
 " a publick examination, been annulled by an express
 " order of the Master-General of the Ordnance.—I
 " could mention some particulars, supported by good
 " authority,

“ authority, that occurred in the course of that ex-
 “ amination, which would but ill agree with the im-
 “ portance he assumes in his confident accusations;
 “ but I do not think it worth while: This gentleman
 “ has, himself, by his different publications, so well
 “ convinced the world of his abilities, as to render
 “ any farther comment on that head intirely unne-
 “ cessary and ineffectual.” And happy it was for the
 Academy that the Board of Ordnance relieved the
 Professor of Mathematicks from any control of that
 of Fortification, considering what sort of professors
 have succeeded Mr. Muller, and probably may again;
 for whatever he might be in other respects, he was not
 altogether unacquainted with the mathematical sciences.

In 1748 came out Mr. Simpson's *Trigonometry, Plane and Spherical, with the Construction and Application of Logarithms*, 8vo. This little book contains several things new and useful.

In 1750 came out, in two volumes, 8vo, *The Doctrine and Application of Fluxions, containing, besides what is common on the Subject, a Number of new Improvements in the Theory, and the Solution of a Variety of new and very interesting Problems in different Branches of the Mathematics*.—In the preface, the author offers this to the world as a new book, rather than a second edition of that which was published in 1737, in which he acknowledges, that, besides errors of the press, there are several obscurities and defects, for want of experience, and the many disadvantages he then laboured under, in his first sally.

The idea and explanation here given of the first principles of Fluxions, are not essentially different from what they are in his former treatise; though expressed in other terms. The consideration of *time* introduced into the general definition, will, he says, perhaps be disliked by those who would have fluxions to be *mere velocities*: but the advantage of considering them otherwise, viz. not as the velocities themselves, but as magnitudes they would uniformly generate in a given

a given time, appears to obviate any objection on that head. By taking fluxions as mere velocities, the imagination is confined as it were to a point, and without proper care insensibly involved in metaphysical difficulties. But according to this other mode of explaining the matter, less caution in the learner is necessary, and the higher orders of fluxions are rendered much more easy and intelligible. Besides, though Sir Isaac Newton defines fluxions to be the velocities of motions, yet he has recourse to the increments or moments generated in equal particles of time, in order to determine those velocities; which he afterwards teaches to expound by finite magnitudes of other kinds. This work was dedicated to GEORGE EARL of MACCLESFIELD.

In 1752 appeared, in 8vo, the *Select Exercises for young Proficients in the Mathematics*. This neat volume contains, A great Variety of algebraical Problems, with their Solutions. A select Number of Geometrical Problems, with their Solutions, both algebraical and geometrical. The Theory of Gunnery, independant of the Conic Sections. A new and very comprehensive Method for finding the Roots of Equations in Numbers. A short Account of the first Principles of Fluxions. Also the Valuation of Annuities for single and joint Lives, with a Set of new Tables, far more extensive than any extant. This last part was designed as a supplement to his Doctrine of Annuities and Reversions; but being thought too small to be published alone, it was inserted here at the end of the *Select Exercises*; from whence however it has been removed in the last editions, and referred to its proper place, the end of the Annuities, as before mentioned. The examples that are given to each problem in this last piece, are according to the London bills of mortality; but the solutions are general, and may be applied with equal facility and advantage to any other table of observations. The volume is dedicated to JOHN BACON, Esq. F. R. S.

Mr.

Mr. Simpson's *Miscellaneous Tracts*, printed in 4to, 1757, was his last legacy to the public: a most valuable bequest, whether we consider the dignity and importance of the subjects, or his sublime and accurate manner of treating them.

The first of these papers is concerned in determining the *Precession of the Equinox, and the different Motions of the Earth's Axis, arising from the Attraction of the Sun and Moon*. It was drawn up about the year 1752, in consequence of another on the same subject, by M. de Sylvabelle, a French gentleman. Though this gentleman had gone through one part of the subject with success and perspicuity, and his conclusions were perfectly conformable to Dr. Bradley's observations; it nevertheless appeared to Mr. Simpson that he had greatly failed in a very material part, and that indeed the only very difficult one; that is, in the determination of the momentary alteration of the position of the earth's axis, caused by the forces of the sun and moon; of which forces, the quantities, but not the effects, are truly investigated. The second paper contains the Investigation of a very exact Method or Rule for finding the Place of a Planet in its Orbit, from a correction of Bishop Ward's circular Hypothesis, by Means of certain Equations applied to the Motion about the upper Focus of the Ellipse. By this Method the Result, even in the Orbit of Mercury, may be found within a Second of the Truth, and that without repeating the Operation. The third shews the Manner of transferring the Motion of a Comet from a parabolic Orbit, to an elliptic one; being of great Use, when the observed Places of a (new) Comet are found to differ sensibly from those computed on the Hypothesis of a parabolic Orbit. The fourth is an Attempt to shew, from mathematical Principles, the Advantage arising from taking the Mean of a Number of Observations, in practical Astronomy; wherein the Odds that the Result in this Way, is more exact than from one single Observation, is evinced

evinced, and the Utility of the Method in Practice, clearly made appear. The fifth contains the Determination of certain Fluents, and the Resolution of some very useful Equations, in the higher Orders of Fluxions, by Means of the Measures of Angles and Ratios, and the right and versed Sines of circular Arcs. The 6th treats of the Resolution of algebraical Equations, by the Method of Surd-divisors; in which the Grounds of that Method, as laid down by Sir Isaac Newton, are investigated and explained. The 7th exhibits the Investigation of a general Rule for the Resolution of Isoperimetrical Problems of all Orders, with some Examples of the Use and Application of the said Rule. The 8th, or last part, comprehends the Resolution of some general and very important Problems in Mechanics and Physical Astronomy; in which, among other Things, the principal Parts of the 3d and 9th Sections of the first Book of Sir Isaac Newton's Principia, are demonstrated in a new and concise Manner. But what may perhaps best recommend this excellent tract, is the application of the general equations, thus derived, to the determination of the Lunar Orbit.

According to what Mr. Simpson had intimated at the conclusion of his Doctrine of Fluxions, the greatest part of this arduous undertaking was drawn up in the year 1750. About that time M. Clairaut, a very eminent mathematician of the French Academy, had started an objection against Sir Isaac Newton's general law of gravitation. This was a motive to induce Mr. Simpson (among some others) to endeavour to discover whether the motion of the moon's apogee, on which that objection had its whole weight and foundation, could not be truly accounted for, without supposing a change in the received law of gravitation, from the inverse ratio of the squares of the distances. The success answered his hopes, and induced him to look farther into other parts of the theory of the moon's motion, than he had at first intended :

intended: but before he had completed his design, M. Clairaut arrived in England, and made Mr. Simpson a visit; from whom he learnt, that he had a little before printed a piece on that subject, a copy of which Mr. Simpson afterwards received as a present, and found in it the same things demonstrated, to which he himself had directed his enquiry, besides several others.

The facility of the method Mr. Simpson fell upon, and the extensiveness of it, will in some measure appear from this, that it not only determines the motion of the apogee, in the same manner, and with the same ease, as the other equations, but utterly excludes all that dangerous kind of terms that had embarrassed the greatest mathematicians, and would, after a great number of revolutions, entirely change the figure of the moon's orbit. From whence this important consequence is derived, that the moon's mean-motion, and the greatest quantities of the several equations, will remain unchanged, unless disturbed by the intervention of some foreign or accidental cause. These tracts are inscribed to the EARL of MACCLESFIELD, President of the Royal Society.

Besides the foregoing, which are the whole of the regular books or treatises that were published by Mr. Simpson, he wrote and composed several other papers and fugitive pieces, as follow.

Several papers of his were read at the meetings of the Royal Society, and printed in their Transactions: but as most, if not all of them, were afterwards inserted, with alterations or additions, in his printed volumes, it is needless to take any farther notice of them here.

He proposed, and resolved many questions in the Ladies Diaries, &c. sometimes under his own name, as in the years 1735 and 1736; and sometimes under feigned or fictitious names; such as, it is thought, Hurlothrumbo, Kubernetes, Patrick O'Cavenah, Mar-maduke Hodgson, Anthony Shallow, Esq. and probably several others; see the Diaries for the years 1735,

1736, 42, 43, 53, 54, 55, 56, 57, 58, 59, and 60. Mr. Simpson was also the editor or compiler of the Diaries from the year 1754 till the year 1760, both inclusive, during which time he raised that work to the greatest degree of respect. He was succeeded in the editorship by Mr. Edw. Rollinson. See my Diarian Miscellany, vol. 3.

It has also been commonly supposed that he was the real editor of, or had a principal share in, two other periodical works of a miscellaneous mathematical nature; viz. the Mathematician, and Turner's Mathematical Exercises, two volumes, in 8vo, which came out in periodical numbers, in the years 1750 and 1751, &c. The latter of these seems especially to have been set on foot to afford a proper place for exposing the errors and absurdities of Mr. Robert Heath, the then conductor of the Ladies Diary and the Palladium; and which controversy between them ended in the disgrace of Mr. Heath, and expulsion from his office of editor to the Ladies Diary, and the substitution of Mr. Simpson in his stead, in the year 1753.

In the year 1760, when the plans proposed for erecting a new bridge at Blackfriars were in agitation, Mr. Simpson, among other gentlemen, was consulted upon the best form for the arches, by the New-bridge Committee. Upon this occasion he gave a preference to the semicircular form; and, besides his report to the Committee, some letters also appeared, by himself and others, on the same subject, in the public newspapers, particularly in the Daily Advertiser, and in Lloyd's Evening Post. And the same were also collected in the Gentleman's Magazine for that year, page 143 and 144.

It is probable that this reference to him, gave occasion to the turning his thoughts more seriously to this subject, so as to form the design of composing a regular treatise upon it: for his family have often informed me that he laboured hard upon this work for some time before his death, and was very anxious to have completed it, frequently remarking to them, that
this

this work, when published, would procure him more credit than any of his former publications. But he lived not to put the finishing hand to it. Whatever he wrote upon this subject, probably fell, together with all his other remaining papers, into the hands of Major Henry Watson, of the Engineers, in the service of the India Company, being in all a large chest full of papers. This gentleman had been a pupil of Mr. Simpson's, and had lodged in his house. After Mr. Simpson's death, Mr. Watson prevailed upon the widow to let him have the papers, promising either to give her a sum of money for them, or else to print and publish them for her benefit. But nothing of the kind was ever done; this gentleman always declaring, when urged on this point by myself and others, that no use could be made of any of the papers, owing to the very imperfect state in which he said they were left. And yet he persisted in his refusal to give them up again.

From Mr. Simpson's writings, I now return to himself. Through the interest and solicitations of the beforementioned William Jones, Esq. he was, in 1743, appointed professor of mathematics, then vacant by the death of Mr. Derham, in the Royal Academy at Woolwich; his warrant bearing date August 25th. And in 1745 he was admitted a fellow of the Royal Society, having been proposed as a candidate by Martin Folkes, Esq. President, William Jones, Esq. Mr. George Graham, and Mr. John Machin, Secretary; all very eminent mathematicians. The president and council, in consideration of his very moderate circumstances, were pleased to excuse his admission fees, and likewise his giving bond for the settled future payments.

At the academy he exerted his faculties to the utmost, in instructing the pupils who were the immediate objects of his duty, as well as others, whom the superior officers of the ordnance permitted to be boarded and lodged in his house. In his manner of teaching he had a peculiar and happy address; a certain dignity and perspicuity, tempered with such a

degree of mildness, as engaged both the attention, esteem and friendship of his scholars; of which the good of the service, as well as of the community, was a necessary consequence.

It must be acknowledged however, that his mildness and easiness of temper, united with a more inactive state of mind, in the latter years of his life, rendered his services less useful; and the same very easy disposition, with an innocent, unsuspecting simplicity, and playfulness of mind, rendered him often the dupe of the little tricks of his pupils. Having discovered that he was fond of listening to little amusing stories, they took care to furnish themselves with a stock; so that, having neglected to learn their lessons perfect, they would get round him in a croud, and, instead of demonstrating a proposition, would amuse him with some comical story, at which he would laugh and shake very heartily, especially if it were tinged with somewhat of the ludicrous or smutty; by which device they would contrive imperceptibly to wear out the hours allotted for instruction, and so avoid the trouble of learning and repeating their lesson. They tell also of various tricks that were practised upon him in consequence of the loss of his memory in a great degree, in the latter stage of his life.

Indeed it may seem strange that a tall, large, and seemingly strong-bodied man (for such was the appearance and figure of Mr. Simpson), should at fifty years old, without sickness, have all his faculties quite worn out, and die of extreme old age. But, notwithstanding his bulk and figure, he was really of no very strong constitution of body, having originally been framed with weak nerves: besides which, there were two other principal causes for his premature decay: the first of these was his very close and uninterrupted application to study, almost continually by night and by day, which produced an extraordinary exhausting of his spirits, without allowing him time to relieve and recruit them by proper air and exercise, and to enjoy the conversation of his friends: the other was the want of proper nourishment, and the cruelty

cruelty of a wasteful wife and family, whose extravagance kept him always poor and drudging to supply their voracious demands, and withheld from him the nourishment proper for his comfortable existence. Indeed the many stories I have been told of her ill temper, and barbarous treatment of a man so gentle, mild and placid, as Mr. Simpson was, are quite shocking, by which his very useful and valuable life was doubtless shortened by many years: while she was revelling with her family and gossips in the parlour, he was banished to the garret, to toil at his book, with no other supper than the thin milk from which she had skimmed the cream for her tea, and accompanied with an order that he should not be seen below for the night. She would break open his cupboard to rob him of his cordial, and pick his pockets of the trifle he had saved to spend with his friends; and numberless other unnatural actions.

It has been said that Mr. Simpson frequented low company, with whom he used to guzzle porter and gin: but it must be observed that the misconduct of his family put it out of his power to keep the company of gentlemen, as well as to procure better liquor.

In the latter stage of his existence, when his life was in danger, exercise and a proper regimen were prescribed him, but to little purpose; for he sunk gradually into such a lowness of spirits, as often in a manner deprived him of his mental faculties, and at last rendered him incapable of performing his duty, or even of reading the letters of his friends; and so trifling an accident as the dropping of a tea-cup would flurly him as much as if a house had tumbled down.

The physicians advised his native air for his recovery; and in February, 1761, he set out, with much reluctance (believing he should never return) for Bosworth, along with some relations. The journey fatigued him to such a degree, that upon his arrival he betook himself to his chamber, where he grew continually worse and worse, to the day of his death, which happened the 14th of May, in the fifty-first year of his age.

Upon

Upon this occasion a reflection forces itself upon me, concerning the insalubrity of the situation and residence of the professor of mathematics at Woolwich, and the similarity of its effects upon the whole succession of professors, Mr. Simpson, Mr. John Lodge Cowley, and myself, who every one of us, each after the other, entered upon our appointments at an early age, and with the best prospects of health and long life, that a vigorous constitution, founded by a virtuous education at a distance from the capital, could promise. Yet, such is the closeness of the professor's apartments, and the almost impossibility of his taking proper air and exercise, joined to the natural sedentary habits of a studious profession, that it has uniformly reduced us all in a similar manner to the brink of the grave, and nearly in the same space of time. Mr. Simpson was gradually reduced by it to a condition too low before he changed his situation, and ended his valuable labours after seventeen years residence. Mr. Cowley, after twelve years, was so reduced, as not to be able to reside and perform his duty any longer; when, being superannuated on a pension, he retired to a situation of better air, and where he could easily have proper exercise; which presently had the best effect in restoring his health, and he is now, after almost twenty years, in as healthy and robust a condition as any man whatever. In like manner, succeeding Mr. Cowley, I began my career with a close application to study, to which I was induced both by inclination, and the want of company and exercise: but soon found an alteration in my health, which gradually declined, till, at the end of thirteen years, I was unable to continue my residence any longer, and obliged to retire to a more elevated situation, at a small distance, from whence, by walking always to the academy and back again, my health and strength gradually improved; and as this experiment evinced the benefit of the change I had been compelled to make, his Grace the Master General of the ordnance, and the principal officers of that board, have been pleased to permit my residence in the like situation
ever

ever since, which is now more than six years, and with the same manifest benefit to my health and activity.

But to return to our Author. At his death, Mr. Simpson left in being, besides his widow, her daughter and son by her first husband, and his own daughter and son, the latter of whom was born after his removal to London. The two daughters had married officers of artillery, and young Simpson had been presented with a commission in the same corps; but all or most of them spending and ending their lives in a manner that might be expected from an education under such a mother: young Simpson lived to have a company in the Royal Artillery; but on account of a besotted habit of drunkenness, and other misconduct, he was broke; and several years after my settlement at Woolwich, he was wandering about as a vagabond, and a scandal to all that knew him: having not heard of him for several years, I suppose he has died a miserable death. The daughters likewise are both dead. As to Mr. Simpson's widow, the King, at the instances of Lord Viscount Ligonier, the then Master General, in consideration of Mr. Simpson's great merits, was graciously pleased to grant her a handsome pension for life, together with apartments adjoining to the academy; a favour that had never been conferred on any before. Here she lived, and her son Swinfield, the taylor, and his wife, along with her, in the same apartments, for many years after my settling at Woolwich; and from one or other of whom I have at different times been informed of the particulars above related, with many others. She died only December the 14th, 1782, at the great age of one hundred and two. Her son Swinfield is still living, and resides in Woolwich, supported by a weekly allowance from some relations, who are in very genteel circumstances; he is now about eighty-four years of age, and looks so well, that he may probably still exist several years longer.

*Royal Academy, Woolwich,
July 28, 1792.*

C. H.

The following BOOKS are all written by Mr. THOMAS SIMPSON, F. R. S. and printed for F. WINGRAVE, Successor to Mr. NOURSE, in the STRAND.

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PART

PART I

CONTAINING

*A select Number of ALGEBRAICAL PROBLEMS,
with their SOLUTIONS.*

DESIGNED

As proper Exercises for young Beginners.

QUESTION I.

WHAT Number is that, which being doubled and 16 added to the Product, the Sum shall be 188?

Let x represent the required number;
then $2x$ will denote the double thereof;
and so $2x + 16 = 188$, by the question.

Therefore $2x = 188 - 16 = 172$, by transposition.

And $x = \frac{172}{2} = 86$, by division.

QUESTION II.

To find that Number, which being added to 56, the Treble of the required Number shall be produced.

If x be put for the number sought,
then $3x$ will be the treble thereof:
and therefore $3x = x + 56$, by the question.
Hence $2x = 56$, by transposition.

And $x = \frac{56}{2} = 28$, by division.

B

QUESTION

QUESTION III.

The Sum of 155l. was raised (for a certain Purpose) by three different Persons, A, B, and C; whereof B advanced 15l. more than A; and C, 20l. more than B: How much did each contribute?

Let x be the number of pounds advanced by A.
 Then $x+15$ is the number of pounds advanced by B,
 and $x+35$ the number of pounds advanced by C.
 Therefore $3x+50=155$, by the question.
 Whence $3x=105$,
 and $x=35$.
 From which it also appears that B contributed 50l. and C 70l.

QUESTION IV.

A Gentleman (by Will) left 550l. to be divided among four Servants, A, B, C, and D; whereof B was to have twice as much as A; C as much as A and B; and D as much as C and B. How much had each?

Let x be the number of A's pounds.
 Then $2x$ is the number of B's pounds,
 also $3x$ is the number of C's pounds;
 and $5x$ the number of D's pounds:
 therefore $11x=550$, by question.

And, consequently, $x=\frac{550}{11}=50$.

From which the rest of the shares are easily determined.

QUESTION V.

It is required to divide the Number 92 into four such Parts, that the first may exceed the second by 10, the third by 18, and the fourth by 24.

Let x be the first part.

Then $\begin{cases} x-10 \\ x-18 \\ x-24 \end{cases}$ will be the other parts.

And

And $4x - 52 = 92$, by the question.

Hence $4x = 144$;

and $x = \frac{144}{4} = 36$.

QUESTION VI.

A certain Sum of Money was shared among five Persons, A, B, C, D, and E; whereof B received 10l. less than A; C 16l. more than B; D 5l. less than C; and E 15l. more than D. Moreover it appeared that E received as much as both A and B. What was the whole Sum shared, and how much did each receive?

Let x be the share of A.

Then $\begin{cases} x - 10 \\ x + 6 \\ x + 1 \\ x + 16 \end{cases}$ will be $\begin{cases} B, \\ C, \\ D, \\ E: \end{cases}$ that of

therefore $x + 16 = 2x - 10$, by the question.

Whence $26 = x$.

From which it appears that 26, 16, 32, 27, and 42 (pounds) were the respective shares; and that the whole sum was 143l.

QUESTION VII.

To find that Number, whereof the Double increased by 24, shall as much exceed 80, as the Number itself is below 100.

Let x be the required number.

Then $2x + 24 - 80 = 100 - x$, by the question.

Whence $2x + x = 100 - 24 + 80$,

that is $3x = 156$;

and therefore $x = \frac{156}{3} = 52$.

ALGEBRAICAL PROBLEMS,

QUESTION VIII.

What two Numbers are those, whereof the Difference is 7 and the Sum 33.

Let x be the lesser number,
then $x+7$ will be the greater,
and $2x+7=33$.

Therefore $2x=33-7=26$,

and $x=\frac{26}{2}=13$ the lesser.

Whence $x+7=20$ the greater.

QUESTION IX.

To divide the Number 75 into two such Parts, that 3 times the greater may exceed 7 times the lesser by 15.

If x be the greater part,

then $75-x$ will be the lesser } *by the question.*

and $3x=75-x \times 7+15$

that is, $3x=75-7x+15$;

therefore $10x=540$, and $x=54$.

From whence the lesser part $(75-x)$ is found $=21$.

QUESTION X.

A, after winning 10 Guineas of B, had as much Money as B and 6 Guineas more; and betwixt them both they had forty Guineas. What Money had each at first?

Let x be the guineas that A began with;
then $40-x$, are the guineas that B began with:
therefore, after play,

A had, $x+10$ guineas;

and B, $30-x$ guineas.

Whence $x+10=30-x+6$ (by the question.)

Therefore $2x=26$.

And $x=13$.

QUESTION

QUESTION XI.

The Sum of 500l. was divided among four Persons, so that the first and second, between them, had 280l; the first and third 260l; and the first and fourth 220l. How many Pounds had Each?

If x be the number of pounds the first had,

then the $\left\{ \begin{array}{l} 2^d \\ 3^d \\ 4^{th} \end{array} \right\}$ had $\left\{ \begin{array}{l} 280-x. \\ 260-x. \\ 220-x. \end{array} \right.$

The sum of all which, being $760-2x$, is $=500$, by the question.

Whence $x = \frac{760-500}{2} = 130$.

Therefore the four shares were 130, 150, 130, and 90 pounds, respectively.

QUESTION XII.

'Tis proposed to divide 60 into two such parts, that the Difference between the Greater and 64, may be equal to twice the Difference between the Lesser and 38.

If x be the greater part;

then $60-x$ will be the lesser:

also $64-x$ will be the first mentioned difference,

and $38-60+x$, or $x-22$, will be the second.

Therefore $64-x = 2 \times x - 22$, by the question.

that is, $64-x = 2x-44$.

Whence $108 = 3x$, and $36 = x$.

QUESTION XIII.

After 34 Gallons had been drawn out of one of two, equal, Casks, and 80 Gallons out of the other, there remained just twice as much Liquor in the one as in the other. What did each Cask contain when full?

Let x be the number of gallons sought;

then $x-34$ will be what remained in the first cask,

and

and $x-80$, what remained in the second,

Hence $x-34=2 \times x-80$, by the question.

Or, $x-34=2x-160$.

Therefore $126=x$.

QUESTION XIV.

A Son asking his Father how old he was, received the following Answer. Your Age four Years ago, says the Father, was only $\frac{1}{4}$ of mine, at that Time; but now your Age is just $\frac{1}{2}$ of mine. What was the Age of each?

Let x be the age of the son,
then $3x$ will be that of the father:
also $x-4$ will be the age of the son four years before
the time in question; and $3x-4$ will be the corresponding age of the father: which, by the question, is equal to 4 times $x-4$. Hence we have this equation
 $4x-16=3x-4$.

Therefore $x=12$, and $3x=36$; which are the two ages required.

QUESTION XV.

What Number is that, whose $\frac{1}{3}$ exceeds its $\frac{1}{4}$ Part by 16?

Let x be the required number; then, its $\frac{1}{3}$ part being $\frac{x}{3}$, and its $\frac{1}{4}$ part $\frac{x}{4}$,

we have $\frac{x}{3}-\frac{x}{4}=16$, by the question.

Hence $4x-3x=192$, by reduction.
that is, $x=192$.

QUESTION

QUESTION XVI.

In a Mixture of Wine and Cyder, one half of the whole +25 Gallons was Wine; and $\frac{1}{3}$ Part -5 Gallons Cyder. How many Gallons were there of each?

If x be put for the number of gallons in the whole mixture, the gallons of wine will be expressed by $\frac{x}{2} + 25$, and those of cyder by $\frac{x}{3} - 5$: which together being equal to the whole, we

therefore have $\frac{x}{2} + 25 + \frac{x}{3} - 5 = x$;

$$\text{or, } 20 = x - \frac{x}{2} - \frac{x}{3} :$$

Hence $120 = 6x - 3x - 2x$;

or $120 = x$. From which it appears that the mixture consisted of 85 gallons of wine, and 35 of cyder.

QUESTION XVII.

In a Lottery consisting of 100000 Tickets, half the Number of Prizes added to $\frac{1}{3}$ of the Number of Blanks, was 35000. How many Prizes were there in the Lottery?

If x be the number of prizes; then $100000 - x$, will be the number of blanks.

$$\text{And so, } \frac{x}{2} + \frac{100000 - x}{3} = 35000.$$

Hence $3x + 200000 - 2x = 210000$:

and $x = 10000 =$ the number sought.

QUESTION

QUESTION XVIII.

To the Composition of a certain Quantity of Gunpowder, $\frac{1}{2}$ of the whole Weight + 6lb. of Saltpetre was necessary; the Sulphur used was $\frac{1}{3}$ of the whole + 5lb. and the Charcoal $\frac{1}{4}$ of the whole - 3lb. How many Pounds of each of the three Ingredients were there taken?

Let x be the number of pounds in the whole: then

$$\begin{array}{l} \text{there} \\ \text{were} \end{array} \left\{ \begin{array}{l} \frac{x}{2} + 6, \text{ pounds of saltpetre.} \\ \frac{x}{3} - 5, \text{ pounds of sulphur.} \\ \frac{x}{4} - 3, \text{ pounds of charcoal.} \end{array} \right.$$

And therefore $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} - 2 = x$, by the question.

Whence $12x + 8x + 6x - 48 = 24x$:

And consequently $x = \frac{48}{2} = 24$.

Therefore there were taken 18lb. of saltpetre, 3lb. of sulphur, and 3lb. of charcoal.

QUESTION XIX.

A General, after having lost a Battle, found that he had only $\frac{1}{2}$ of his Army + 3600 left, fit for Action; $\frac{1}{8}$ of his Men + 600 being wounded, and the Rest, which were $\frac{1}{5}$ of the whole Army, either slain, taken Prisoners, or missing. What was the whole Number of the Army?

The number fought being denoted by x , the number of men left unhurt will be $\frac{x}{2} + 3600$.

And the number of the wounded $\frac{x}{8} + 600$.

To which adding, $\frac{x}{5}$, the number of the slain and prisoners, we have the number of the whole army;

or

$$\text{or } \frac{x}{2} + 3600 + \frac{x}{8} + 600 + \frac{x}{5} = x.$$

$$\text{Hence } 4200 (=x - \frac{x}{2} - \frac{x}{8} - \frac{x}{5}) = \frac{3x}{8} - \frac{x}{5}.$$

$$\text{Therefore } 168000 (=15x - 8x) = 7x : \\ \text{and } 24000 = x.$$

QUESTION XX.

A Prize of 2000l. was divided between two Persons, A and B; whose Shares therein were in proportion as 7 to 9. What was the Share of Each?

Let x denote the share of A;
then $2000 - x$, will be that of B.

But $x : 2000 - x :: 7 : 9$, by the question.

From whence (as the rectangle of the two extremes, of any four proportional numbers, is equal to the rectangle of the two means) we get this equation,

$$9 \times x = 2000 - x \times 7;$$

$$\text{that is, } 9x = 14000 - 7x.$$

$$\text{Hence } 16x = 14000,$$

$$\text{and } x = 875.$$

Therefore the share of A was 875*l.* and that of B, 1125*l.*

QUESTION XXI.

To divide 44 into two such Parts, that the Greater increased by 5, may be to the Lesser increased by 7, as 4 is to 3.

If x be the greater part,

$44 - x$, will be the lesser,

and $x + 5 : 51 - x :: 4 : 3$, by the question.

Therefore (by multiplying extremes and means) we have

$$3x + 15 = 204 - 4x :$$

$$\text{Whence } x = \frac{204 - 15}{7} = \frac{189}{7} = 27.$$

C

QUESTION

QUESTION XXII.

To find two Numbers in the proportion of 1 to 2 ; so that, 12 being added to Each, the Sums shall be in proportion as 5 is to 7.

Let x be the lesser number ;

then $2x$, will be the greater.

Hence $x+12 : 2x+12 :: 5 : 7$, by the question.

Therefore $2x+12 \times 5 = x+12 \times 7$;

that is $10x+60=7x+84$.

From which $3x=24$; and $x=8$.

So that the two numbers are 8 and 16.

QUESTION XXIII.

A, at Play, first won 5 Guineas of B, and had then as much Money as B ; but B, upon winning back his own Money and 5 Guineas more, had 5 times as much Money as A. What Money had each at first ?

If x be the number of A's guineas, at first ; then it is plain, from the question, that $x+10$ will be the number of B's guineas.

Whence $x+10+5 (=x-5 \times 5) = 5x-25$,

that is, $40=4x$.

Therefore $x=10$ = the number of A's guineas ;

and $x+10=20$ = the number of B's guineas.

QUESTION XXIV.

A Grocer, with 56lb. of fine Tea, at 20 Shillings a Pound, would mix a coarser Sort, of 14 Shillings, so as to afford the whole, together, at 18 Shillings per Pound. What Quantity of the latter Sort must he take ?

Let x be the number of pounds required ;

then 1120 is the value of the finest sort,

and $14x$ that of the coarsest.

Moreover,

Moreover, the number of pounds of both sorts together being $56+x$, it is evident that $56+x \times 18$, or $1008+18x$, is the value of the whole mixture. And therefore $1120+14x=1008+18x$. Whence $112=4x$. And consequently $28=x$.

QUESTION XXV.

A Farmer would mix Wheat, at 4 Shillings a Bushel, with Rye, at 2s. 6d. a Bushel; so that the whole Mixture may consist of 90 Bushels, and be afforded at 3s. 2d. a Bushel. It is required to find how many Bushels of each Sort must be taken.

If the number of bushels of wheat be x , Those of rye will be $90-x$. Moreover, the value of the wheat will be $48x$, pence, and the value of the rye $90-x \times 30$, pence. Whence $48x+90-x \times 30=90 \times 38$ (by the question.) that is, $48x+2700-30x=3420$: therefore $18x=720$: and consequently $x=\frac{720}{18}=40$.

QUESTION XXVI.

A Workman was hired for 40 Days, at 3s. 4d. per Day, for every Day he worked; but with this Condition, that for every Day he played, he was to forfeit 1s. 4d: and it so happened, that, upon the whole, he had 3l. 3s. 4d. to receive. The question is, to find how many Days of the 40 he worked, and how many he played.

Let x be the number of days he worked; then $40-x$ will be the number of days he played. Moreover, as he was to receive 40 pence, for every day he worked, and to forfeit 16 pence for every day he played,

we have $40x$ = the number of pence earned by work,
 and $40 - x \times 16$ = the number of pence forfeited by play :
 whence $40x - 40 - x \times 16 = 760$, *by the question* ;
 that is, $40x - 640 + 16x = 760$:
 therefore $56x = 1400$;
 and consequently $x = 25$.
 From which it is evident that he worked 25 days, and
 played 15.

QUESTION XXVII.

*A Bill of 120l. was paid in Guineas and Moidores, and
 the Number of Pieces used of both Sorts was just 100.
 How many were there of each ?*

If x be put for the number of guineas ;
 then $100 - x$ will be the number of moidores.
 And so, $21x + 100 - x \times 27 = 120 \times 20$ } *by the question.*
 or, $21x + 2700 - 27x = 2400$.
 Hence $300 = 27x - 21x = 6x$.
 And consequently $x = \frac{300}{6} = 50$.

QUESTION XXVIII.

*One bought 30 Pounds of Sugar, of two different Sorts,
 and paid for the whole 19 Shillings; the best Sort cost
 10d per Pound, and the worst 7d. How many Pounds
 were there of each ?*

Let x stand for the number of pounds of the best sort,
 and then $30 - x$ will express the pounds of the other sort.
 Therefore $x \times 10 + 30 - x \times 7 = 19 \times 12$, *by the question* ;
 that is, $10x + 210 - 7x = 228$.
 Whence $3x = 228 - 210 = 18$,
 and $x = 6$. Therefore there were 6 pounds of the best
 sort, and 24 of the worst.

QUESTION

QUESTION XXIX.

A Lady gave a Guinea in Charity, among a Number of Poor, consisting of Men, Women, and Children: each Man had 12d. each Woman 6d. and each Child 3d. Moreover there were twice as many Women as Men, wanting 2; and 3 times as many Children as Women, wanting 4. How many Persons were there relieved?

Let x be the number of the men;
then $2x-2$, will be the number of women,
and $6x-10$, the number of children.

Hence $x \times 12 + 2x - 2 \times 6 + 6x - 10 \times 3 = 21 \times 12$;

that is, $12x + 12x - 12 + 18x - 30 = 252$,

or, $42x = 294$.

Whence $x = 7$.

Therefore there were 7 men, 12 women, and 32 children; in all 51 persons.

QUESTION XXX.

To find that Number, which being divided, either into three or four equal Parts, the continual Product of all the Parts, in both Cases, shall be exactly the same.

Let x be the required number;
so shall the continual product of the three equal parts

be $\frac{x}{3} \times \frac{x}{3} \times \frac{x}{3} = \frac{x^3}{27}$; and that of the four equal

parts $\frac{x}{4} \times \frac{x}{4} \times \frac{x}{4} \times \frac{x}{4} = \frac{x^4}{256}$.

Whence $\frac{x^4}{256} = \frac{x^3}{27}$, by the question.

Therefore $27x = 256$; and $x = 9\frac{13}{27}$.

QUESTION

QUESTION XXXI.

To find two Numbers, in the Proportion of 3 to 4, whose Sum is to the Sum of their Squares, as 7 to 50.

Let $3x$ denote the lesser number :

then $4x$ will express the greater.

And we shall have $3x+4x : 9x^2+16x^2 :: 7 : 50$,

or, $7x : 25x^2 :: 7 : 50$, by the question.

Therefore $25x^2 \times 7 = 7x \times 50$,

or, $25x^2 = 50x$;

whence $x = \frac{50}{25} = 2$. So that 6 and 8 are the two num-

bers that answer the question.

QUESTION XXXII.

To find two Numbers in the Proportion of 9 to 7 ; so that the Square of their Sum, and the Cube of their Difference, shall be equal.

If $9x$ be put for the greater number ;

then $7x$ will be the lesser ;

And so $\overline{16x}^2 = \overline{2x}^3$, by the question,

that is, $256x^2 = 8x^3$.

Hence $x = \frac{256}{8} = 32$.

Therefore 288 and 224, are the two numbers sought.

QUESTION XXXIII.

To find two Numbers whose Difference is 4 and the Difference of their Squares 120.

Let x be the lesser number ;

then $x+4$, will be the greater :

also xx will be the square of the lesser,

and $xx+8x+16$ that of the greater.

Whence $8x+16=120$, by the question.

Therefore $8x=104$: and $x=13$.

So that 13 and 17 are the two numbers that were to be found.

QUESTION

QUESTION XXXIV.

To divide 100 into two such Parts, that the Difference of their Squares may be 1000.

If x be the greater part,
 $100-x$, will be the lesser.

Therefore $xx - 100-x)^2 = 1000$;

that is, $x^2 - 10000 + 200x - x^2 = 1000$.

Whence $200x = 11000$;

and consequently $x = \frac{11000}{200} = 55$.

QUESTION XXXV.

To divide 100 into two Parts, so that the Square of their Difference may exceed the Square of twice the lesser Part by 2000.

The lesser part being denoted by x ,
 the greater will be expressed by $100-x$,
 and the difference by $100-2x$.

Therefore, by the problem, $(100-2x)^2 = 2x^2 + 2000$,

that is, $10000 - 400x + 4xx = 2xx + 2000$,

or, $10000 - 2000 = 400x$.

Hence $x = \frac{8000}{400} = 20$; and $100-x = 80$.

QUESTION XXXVI.

A and B make a joint Stock of 500l. by which they gain 160l. whereof A, for his Share, had 32l. more than B: What did each Person bring into Stock?

If x be the number of pounds advanced by A;
 then it will be, as 500 (the whole stock) is to 160 (the whole gain) so is x (the stock of A) to $\frac{160x}{500}$, the gain of A: whence the gain of B being $160 - \frac{16x}{50}$:
 we

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we have $\frac{16x}{50} = 160 - \frac{16x}{50} + 32$, by the question

Therefore $\frac{32x}{50} = 192$, or $\frac{x}{50} = 6$;

and consequently $x = 50 \times 6 = 300$.

QUESTION XXXVII.

A Sum of Money was divided between two Persons, A and B, so that the Share of A was to that of B, as 5 to 3, and exceeded $\frac{1}{3}$ of the whole Sum by 50l. What was the Share of each Person?

Let $5x$ express the share of A,

and $3x$ the share of B;

then $8x$ will be the whole sum,

and $\frac{40x}{9}$ will be $\frac{5}{9}$ thereof.

Therefore $5x - \frac{40x}{9} = 50$, by the question.

that is, $\frac{5x}{9} = 50$, or $x = 90$.

Hence 450l. and 270l. are the two shares required.

QUESTION XXXVIII.

A and B began to play together with equal Sums of Money; A first won 20 Guineas, but afterwards lost back the Half of all he then had; and thereupon had only half as much Money as B. What Money did each begin with?

Let x be the number of guineas required:

then A, after winning 20 guineas, had $x + 20$; the

half of which, or $\frac{x}{2} + 10$, is therefore what he had at

last: and this deducted from $2x$ (the whole sum betwixt

both) leaves $2x - \frac{x}{2} - 10 =$ what B had at last.

Therefore

Therefore $2x - \frac{x}{2} - 10 = x + 20$;

whence $3x - 20 = 2x + 40$,

and consequently $x = 60$.

QUESTION XXXIX.

*A Gentleman left his whole Estate among his four Sons; whereof the Eldest had $\frac{1}{2}$ wanting 800*l.* the Second $\frac{1}{4}$ and 120*l.* over; the Third had half as much as the Eldest; and the Youngest $\frac{1}{6}$ of what the Second had. What was the whole Estate? and how much had each?*

Let x be the whole estate; then

The first had $\frac{x}{2} - 800$.

The Second $\frac{x}{4} + 120$.

The third $\frac{x}{4} - 400$.

The fourth $\frac{2x}{12} + 80$.

The sum of all which is, consequently, equal to the whole,

or, $\frac{x}{2} + \frac{x}{4} + \frac{x}{4} + \frac{x}{6} - 1000 = x$.

But it is plain (without reduction)

that $\frac{x}{2} + \frac{x}{4} + \frac{x}{4} = x$.

Hence our equation becomes

$\frac{x}{6} - 1000 = 0$, or $\frac{x}{6} = 1000$.

Therefore the whole estate was 6000*l.* whereof the eldest son had 2200*l.* the second 1620*l.* the third 1100*l.* and the youngest 1080*l.*

QUESTION XL.

One being asked his Age, replied; If $\frac{2}{5}$ of my Years be multiplied by 3, and $\frac{1}{3}$ of them be added to the Product, the Amount will be 115. What was his Age?

If x be the required number of years,

then $\frac{2x}{5} \times 3 + \frac{x}{3} = 115$, by the question;

that is, $\frac{6x}{5} + \frac{x}{3} = 115$.

Therefore $18x + 5x = 15 \times 115$,

or $23x = 15 \times 115$.

Consequently $x = \frac{15 \times 115}{23} = 15 \times 5 = 75$.

QUESTION XLI.

A Person being asked the Hour of the Day, answered thus:

If $\frac{3}{4}$ of the Number of Hours remaining till Midnight be multiplied by 4, the Product will as much exceed 12 Hours, as Half the present Hour from Noon is below 4. What was the Hour after Noon?

Let x be the required hour;

then $12 - x$ will be the hours till midnight,

and $\frac{36-3x}{8} \times 4 - 12 = 4 - \frac{x}{2}$, by the question.

That is, $\frac{36-3x}{2} - 12 = 4 - \frac{x}{2}$,

whence $36 - 3x - 24 = 8 - x$;

and consequently $x = 2$.

QUESTION

QUESTION XLII.

A Market-woman bought in a certain Number of Eggs, at the Rate of 5 for two Pence; one half of which she sold out again at 2 a Penny, and the remaining Half at 3 a Penny; and cleared 4 Pence by so doing. What Number of Eggs had she?

Let $2x$ be the number sought;
then, by the question.

$5 : 2 :: 2x : \frac{4x}{5}$ the number of pence the eggs cost.

But the number of pence they were sold for, again,
is $\frac{x}{2} + \frac{x}{3}$: therefore we have this equation,

$$\text{viz. } \frac{x}{2} + \frac{x}{3} - \frac{4x}{5} = 4.$$

From whence $15x + 10x - 24x = 120$, or $x = 120$.

QUESTION XLIII.

A certain Sum of Money, put out at Interest, amounts, in 8 Months, to 297l. 12s. And, in 15 Months its Amount (computed according to simple Interest) is 306l. What is that Sum? And what the Rate of Interest?

Let x be the number of pounds in the required sum:
then, the interest thereof for 8 months being $297,6 - x$,
and for 15 months $306 - x$, we have,

as $8 : 15 :: 297,6 - x : 306 - x$.

Whence, by multiplying extremes and means,

we get $2448 - 8x = 4464 - 15x$.

Therefore $7x = 2016$,

and consequently $x = 288$ l. the sum required.

For the rate of interest, it will be,

as $288 \text{ l.} \times 15 : 100 \text{ l.} \times 12 :: 18 \text{ l.}$ (the interest of 288l. for 15 months) to 5l. the required interest of 100l. for 12 months.

QUESTION XLIV.

A Waterman finds by Experience, that he can, with the Advantage of a common Tide, row from London to Greenwich, which is 5 Miles, in 3 Quarters of an Hour; and that, to return to London, against an equal Tide, though he rows back along shore, where the Stream is only half as strong as in the Middle, takes him a full Hour and Half. 'Tis required to find, from hence, at what rate per Hour, the tide runs in the Middle where it is strongest.

In the first place it will be

$$\begin{array}{cccc} \text{gr.} & \text{qr.} & \text{m.} & \text{m.} \\ 3 & : 4 & :: 5 & : 6\frac{3}{4} = \text{dist. rowed per hour with the tide,} \end{array}$$

$$6 : 4 :: 5 : 3\frac{1}{2} = \text{dist. rowed per hour against tide.}$$

If now the former of these two distances ($6\frac{3}{4}$) be put $= a$, and the latter ($3\frac{1}{2}$) $= b$; and x be assumed to express the required distance run, per hour, by the stream in the middle of the river; then $a - x$ will be the real effect of his rowing, per hour, in going from London, the motion of the tide being deducted; and $b + \frac{x}{2}$ will be the like effect in his return:

and so, these two quantities being equal to each other, we have $b + \frac{x}{2} = a - x$.

$$\text{Whence } 2b + x = 2a - 2x;$$

$$\text{and consequently } x = \frac{2a - 2b}{3} = 2\frac{1}{3}.$$

QUESTION XLV.

To divide 36 (2) into 3 such Parts, that $\frac{1}{2}$ of the First, $\frac{1}{3}$ of the Second, and $\frac{1}{4}$ of the Third, may be equal to each other.

If x be put for the first part, then is $\frac{x}{2} = \frac{1}{3}$ of the second part (by the question).

And

And so $\frac{3x}{2}$ = second part.

Moreover $\frac{x}{2}$ being $=\frac{1}{4}$ of the third part,

therefore $\frac{4x}{2}$ (or $2x$) = third part.

Hence $x + \frac{3x}{2} + 2x = a$,

and $2x + 3x + 4x = 2a$.

Consequently $x = \frac{2a}{9} = \frac{2 \times 36}{9} = 2 \times 4 = 8$.

From which the second part $\left(\frac{3x}{2}\right)$ appears to be $=12$, and the third $(2x) = 16$.

QUESTION XLVI.

To divide the Number 90 into 4 such Parts, that, if the first be increased by 5, the second diminished by 4, the third multiplied by 3, and the fourth divided by 2, the Result, in each Case, shall be exactly the same.

Let x be the fourth, or last, part:

then, three times the third part being $=\frac{x}{2}$,

the third part will be

Moreover, the second part -4 being, also, $=\frac{x}{2}$,

the second part will be $\frac{x}{2} + 4$:

and, the first part $+5$ being $=\frac{x}{2}$,

the first part, alone, will be $\frac{x}{2} - 5$.

And, by adding all the parts thus found together,

we have $x + \frac{x}{2} + \frac{x}{2} + 4 + \frac{x}{2} - 5 = 90$:

that

that is, $2x + \frac{x}{6} - 1 = 90$.

Whence $13x = 91 \times 6$: and $x = 42$.

Therefore the four required parts are 16, 25, 7, and 42. ~~24~~, respectively.

QUESTION XLVII.

Two Workmen, A and B, were employed together for 50 Days, at 5 Shillings per Day, each; during which Time A, by spending only Sixpence a Day less than B, had saved twice as much as B, besides the Expence of 2 Days over. What did each Person expend a Day?

Let x be the pence A spent per day;
then $60 - x$, will be what he saved per day,
and $54 - x$, what B saved.

Therefore $3000 - 50x$ are A's whole savings,
and $2700 - 50x$ those of B.

Hence $3000 - 50x = 2 \times 2700 - 50x + 2x$;

or, $3000 - 50x = 5400 - 98x$.

From which $48x = 2400$, and $x = 50$.

QUESTION XLVIII.

Two Persons, A and B, have both the same Income; A lays by $\frac{1}{5}$ of his; but B, by spending 60l. per Ann. more than A, at the End of three Years finds himself 100l. in debt. What did each receive, and expend, per Annum?

Let x be the yearly income of each;
then $\frac{4x}{5}$ is the sum expended by A, per ann.

and $\frac{4x}{5} + 60$, that expended by B.

Therefore $\frac{4x}{5} + 60 - x$, is what B runs in debt.

Consequently $\frac{4x}{5} + 60 - x \times 3 = 100$,

or $\frac{12x}{5} + 180 - 3x = 100;$

that is, $180 - \frac{3x}{5} = 100.$

Whence $900 - 3x = 500,$

and $x = \frac{400}{3} = 133\frac{1}{3}$ 6s. 8d.

Therefore A expended 106*l.* 13*s.* 4*d.* and B 166*l.* 13*s.* 4*d.* per annum.

QUESTION XLIX.

A Grazier bought in as many Sheep, of different Sorts, as cost him 33*l.* 7*s.* 6*d.* For the first Sort, which were $\frac{1}{3}$ of the whole, he paid 9*s.* 6*d.* a-piece; for the second Sort, which were $\frac{1}{4}$ of the whole, he paid 11*s.* each; and for the rest, 12*s.* 6*d.* each. What Number of Sheep did he buy in all?

If x be the whole number of sheep;

then, the number of the first sort being $\frac{x}{3}$, and of the second sort $\frac{x}{4}$, the number of the remaining sort (at 12*s.*

6*d.* each) must be $x - \frac{x}{3} - \frac{x}{4} = \frac{12x - 4x - 3x}{12} = \frac{5x}{12}.$

whence, by the conditions of the problem, we have

$\frac{x}{3} \times 19 + \frac{x}{4} \times 22 + \frac{5x}{12} \times 25 = 1335;$

that is, $\frac{19x}{3} + \frac{22x}{4} + \frac{125x}{12} = 1335.$

Let each term of this equation be now multiplied by 12, and it will become

$76x + 66x + 125x = 16020,$

or, $267x = 16020.$

Therefore $x = 60.$

QUESTION

QUESTION L.

A Draper, of a Piece of Cloth, standing him in 3s. 2d. per Yard, sold $\frac{1}{3}$ Part, at 4s. per Yard; $\frac{1}{4}$ at 3s. 8d. per Yard; $\frac{1}{5}$ at 3s. 6d. per Yard; and the Remnant at 3s. 4d. a Yard: and his Gain upon the Whole, was 15s. 2d. How many Yards did the Piece contain?

If the number sought be denoted by x ;
then the number of yards in the remnant

will be $x - \frac{x}{3} - \frac{x}{4} - \frac{x}{5} = \frac{60x - 20x - 15x - 12x}{60} = \frac{13x}{60}$.

Therefore by the question, we have

$$\frac{x}{3} \times 48 + \frac{x}{4} \times 44 + \frac{x}{5} \times 42 + \frac{13x}{60} \times 40 - 38x = 182,$$

$$\text{or, } 16x + 11x + \frac{42x}{5} + \frac{26x}{3} - 38x = 182,$$

$$\text{that is } \frac{42x}{5} + \frac{26x}{3} - 11x = 182:$$

$$\text{whence } 126x + 130x - 165x = 2730.$$

$$\text{And therefore } x = \frac{2730}{91} = 30.$$

QUESTION LI.

A Distiller proposes to mix Foreign Brandy, standing him in 8 Shillings a Gallon, with British Spirits of 3 Shillings per Gallon, in such Proportion that he may gain 30 per Cent by selling out the Compound at 9s. a Gallon. What is that Proportion?

Suppose, that, with a Gallons of Brandy, he mixes x gallons of spirits; then, the brandy, standing him in $8a$ (shillings) and the spirits in $3x$ (shillings); the true value of the whole mixture will be $8a + 3x$.

But the value of $a + x$ gallons, at 9 shillings per gallon, is $9a + 9x$: therefore, by laying out $8a + 3x$, he gains $a + 6x$: and so

we have $8a + 3x : a + 6x :: 100 : 30$, by the question.
Consequently

Consequently $100a + 600x = 240a + 90x$;

whence $510x = 140a$,

and $x = \frac{14a}{51}$.

From which it appears, that to every 51 gallons of brandy, there must be taken 14 gallons of spirits.

QUESTION LII.

To find two Numbers in the Proportion of 4 to 5, from which two other (required) Numbers, in the Proportion of 6 to 7 being, respectively, deducted, the Remainders shall be in the Proportion of 2 to 3, and their Sum equal to 20.

Let $4x$ and $5x$ be the two first numbers, and $6y$ and $7y$ the other 2 numbers.

Then $4x - 6y : 5x - 7y :: 2 : 3$ } by the question.

And $9x - 13y = 20$

From which proportion, by multiplying extremes and means, we have $12x - 18y = 10x - 14y$, and therefore $x = 2y$; which substituted in the above equation gives $18y - 13y = 20$;

whence $y = \frac{20}{5} = 4$; and $x (=2y) = 8$.

Therefore the 2 first numbers are 32 and 40; and the other two, 24 and 28.

QUESTION LIII.

A Farmer sold, at one time, 30 Bushels of Wheat and 40 of Barley, and for the whole received 13*l*. 10*s*.; and, at another time, he sold 50 Bushels of Wheat and 30 of Barley, at the same Prices as before, and for the whole received 17*l*. The Question is, to find what each Sort of Grain was sold at per Bushel

Let x and y express the numbers of shillings, respectively, that the wheat and barley were sold at per bushel; and then, from the conditions of the question, we shall have the two following equations, viz.

E

30x

$$30x + 40y = 270,$$

$$50x + 30y = 340.$$

From 4 times the second of which equations let 3 times the first be subtracted, and there will remain

$$110x = 550.$$

$$\text{Therefore } x = \frac{55}{11} = 5 :$$

$$\text{and } y \left(= \frac{27 - 3x}{4} \right) = 3.$$

QUESTION LIV.

A Farmer, with 28 Bushels of Barley, at 2s. 4d. per Bushel, would mix Rye, at 3s. per Bushel, and Wheat, at 4s. per Bushel; so that the whole Mixture may consist of 100 Bushels, and be worth 3s. 4d. a Bushel. How many Bushels of Rye, and how many of Wheat must be mingled with the Barley?

Let x be the number of bushels of rye, and y those of the wheat: then, the value of the barley being 784 (pence), of the rye $36x$ (pence), and of the wheat $48y$ (pence), we have

$$784 + 36x + 48y = 4000 \quad \left. \vphantom{\begin{array}{l} 784 + 36x + 48y = 4000 \\ 28 + x + y = 100 \end{array}} \right\} \text{by the question.}$$

$$\text{and } 28 + x + y = 100$$

From the first of which equations, take 36 times the second, and there results,

$$784 - 36 \times 28 + 12y = 400,$$

$$\text{that is, } -224 + 12y = 400.$$

$$\text{Therefore } 12y = 624,$$

$$\text{and consequently } y = \frac{624}{12} = 52.$$

$$\text{Whence } x (= 100 - 28 - y) = 20.$$

QUESTION

QUESTION LV.

A and B, working together on the same Work, can earn 40 Shillings in 6 Days; A and C together can earn 54 Shillings in 9 Days; and B and C, 80 Shillings in 15 Days. It is required to find what each Person, alone, can earn per Day.

Let x , y and z express the numbers of shillings in the three required values, respectively.

Then $\begin{cases} 6x+6y=40 \\ 9x+9z=54 \\ 15y+15z=80 \end{cases}$ by the question.

And $\begin{cases} x+y=6\frac{2}{3} \\ x+z=6 \\ y+z=5\frac{1}{3} \end{cases}$ by division.

Hence $y-z=\frac{2}{3}$, by subtracting the 2d equation from 1st.

And $2y=6$, by adding the two last.

Consequently $y=3$.

From which we have $x (=6\frac{2}{3}-y) = 3\frac{2}{3} = 3s. 8d.$ and $z (=5\frac{1}{3}-y) = 2\frac{2}{3} = 2s. 4d.$

QUESTION LVI.

To find three Numbers, so that $\frac{1}{2}$ the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third, shall together be equal to 62; also $\frac{1}{3}$ of the first, $\frac{1}{4}$ of the second, and $\frac{1}{5}$ of the third, equal to 47; and, lastly, $\frac{1}{4}$ of the first, $\frac{1}{5}$ of the second, and $\frac{1}{6}$ of the third, equal to 38.

Put $a=62$, $b=47$, and $c=38$; and let the three required numbers be denoted by x , y and z , respectively; then the conditions of the problem will be expressed in the three following equations, viz.

$$\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = a.$$

$$\frac{x}{3} + \frac{y}{4} + \frac{z}{5} = b,$$

$$\frac{x}{4} + \frac{y}{5} + \frac{z}{6} = c.$$

E 2

Which,

28 ALGEBRAICAL PROBLEMS.

Which, cleared of fractions, become

$$12x + 8y + 6z = 24a$$

$$20x + 15y + 12z = 60b$$

$$30x + 24y + 20z = 120c.$$

Now (in order to exterminate z) let the second of these equations be taken from the double of the first, and also the treble of the third from the quintuple of the second; and there results

$$4x + y = 48a - 60b; \text{ and}$$

$$10x + 3y = 300b - 360c;$$

whence, by deducting the second of these from the treble of the former, and dividing by 2, there comes out

$$x = 72a - 240b + 180c = 24.$$

From which y ($= 48a - 60b - 4x$) is also found $= 60$, and z ($= a - \frac{1}{2}x - \frac{1}{3}y \times 4$) $= 120$.

QUESTION LVII.

To divide the Number 90 (a) into three Parts, so that, the Double of the first Part $+ 40$ (b); the Treble of the second $+ 20$ (c); and the Quadruple of the third $+ 10$ (d), may be all equal to one another.

Let x , y , and z represent the three required parts, respectively; then, from the conditions of the problem, we shall have

$$x + y + z = a,$$

$$2x + b = 3y + c,$$

$$2x + b = 4z + d.$$

Now, in order to exterminate y and z , let 12 times the first equation, 4 times the second, and 3 times the third, be added all together; and you will have

$$26x + 12y + 12z + 7b = 12a + 12y + 12z + 4c + 3d.$$

$$\text{Therefore } 26x = 12a + 4c + 3d - 7b,$$

$$\text{and } x = \frac{12a + 4c + 3d - 7b}{26} = 35.$$

$$\text{Whence } y \left(= \frac{2x + b - c}{3} \right) = 30, \text{ and } z (= a - x - y) = 25.$$

QUESTION

QUESTION LVIII:

To find three Numbers, so that the first with half the other two, the second with $\frac{1}{3}$ of the other two, and the third with $\frac{1}{4}$ of the other two, may be the same, and amounts to 51 in each Case.

Put $a=51$, and let x , y and z denote the three required numbers; then, by the question,

$$x + \frac{y+z}{2} = a,$$

$$y + \frac{x+z}{3} = a,$$

$$z + \frac{x+y}{4} = a.$$

Which, cleared of fractions, become

$$2x + y + z = 2a,$$

$$x + 3y + z = 3a,$$

$$x + y + 4z = 4a.$$

From whence, by taking the second from the third, and the first from the double of the second, there results $-2y + 3z = a$, and $5y + z = 4a$.

And, by deducting the former of these, from the treble of the latter, we have $17y = 11a$,

$$\text{Therefore } y = \frac{11a}{17} = 33,$$

$$z (=4a - 5y) = \frac{13a}{17} = 39,$$

$$\text{and } x (=3a - 3y - z) = \frac{5a}{17} = 15.$$

QUESTION

QUESTION LIX.

A certain Sum of Money was divided between three Persons, A, B and C; so that A's Share exceeded $\frac{1}{4}$ of the Shares of B and C by 30l; also the Share of B exceeded $\frac{1}{3}$ of the Shares of A and C by 30l; and the Share of C likewise exceeded $\frac{1}{3}$ of the Shares of A and B, by 30l. The Question is, to find the Share of each Person.

Let $a=30$; and let x , y , and z be assumed to express the three required numbers; then by the conditions of the problem,

$$x - \frac{4y + 4z}{7} = a,$$

$$y - \frac{3x + 3z}{8} = a,$$

$$z - \frac{2x + 2y}{9} = a.$$

Whence, by reduction,

$$7x - 4y - 4z = 7a,$$

$$-3x + 8y - 3z = 8a,$$

$$-2x - 2y + 9z = 9a.$$

Now, to get rid of y (which, because of the even coefficients, is easiest to be exterminated) let the double of the first equation and the quadruple of the third be, successively, added to the second; by means whereof we have

$$11x - 11z = 22a,$$

$$-11x + 33z = 44a.$$

Moreover, by adding these two last equations together, we have $22z = 66a$,

Therefore $z = 3a = 90$;

whence $x (= 2a + z) = 5a = 150$,

and $y (= a + \frac{3}{8} \times x + z) = 4a = 120$.

QUESTION

QUESTION LX.

If A and B together can perform a Piece of Work in 8 Days; A and C together in 9 Days; and B and C in 10 Days. How many Days will it take each Person, alone, to perform the same Work?

Let the whole work be represented by a , and let x , y , and z stand for the parts thereof performed by A, B, and C in one day, respectively.

then $\begin{cases} 8x+8y=a \\ 9x+9z=a \\ 10y+10z=a \end{cases}$ by the question.

And $\begin{cases} x+y=\frac{a}{8} \\ x+z=\frac{a}{9} \\ y+z=\frac{a}{10} \end{cases}$ by division.

Whence $y-z \left(= \frac{a}{8} - \frac{a}{9} \right) = \frac{a}{72}$,

and $2y = \frac{a}{10} + \frac{a}{72} = \frac{82a}{720}$.

Consequently $y = \frac{41a}{720}$: from which $x \left(= \frac{a}{8} - y \right)$ is found $= \frac{49a}{720}$,

and $z \left(= \frac{a}{10} - y \right) = \frac{31a}{720}$.

Now, the part of the work (a) performed by each person in one single day being thus assigned, the number of days it will take any one of them to do the whole, will be found by dividing the whole by the assigned part.

Thus, $\left(\frac{41a}{720} \right) a = \frac{720a}{41a} = \frac{720}{41} = 17\frac{23}{41}$ is the number of days in which B, alone, can do the whole. And, in like

like manner, the number of days in which A, or C, can do the whole, appears to be

$14\frac{34}{49}$, or $23\frac{7}{31}$, respectively.

QUESTION LXI.

If A, B and C can, together, finish a Piece of Work in 9 Days; A, B and D together, in 10 Days; A, C and D together, in 11 Days; and B, C and D in 12 Days. In how long Time can they all four, together, finish it?

Here, denoting the given numbers by $a, b, c,$ and d , and putting $u, x, y,$ and z , for the parts of the whole work (g) done by each in one day, respectively, we shall, by the question, have these equations,

$$\text{viz. } \begin{cases} a \times u + x + y = g \\ b \times u + x + z = g \\ c \times u + y + z = g \\ d \times x + y + z = g \end{cases}$$

$$\text{or } \begin{cases} u + x + y = \frac{g}{a} \\ u + x + z = \frac{g}{b} \\ x + y + z = \frac{g}{c} \\ x + y + z = \frac{g}{d} \end{cases} \text{ by division.}$$

The sum of all which, divided by 3,

gives $u + x + y + z = \frac{1}{3} \times \frac{g}{a} + \frac{g}{b} + \frac{g}{c} + \frac{g}{d}$;

Therefore seeing the work done by all the four, in one day, is expressed by $\frac{1}{3} \times \frac{g}{a} + \frac{g}{b} + \frac{g}{c} + \frac{g}{d}$, the whole work g , divided hereby, will consequently give

give $\frac{38}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}} = \frac{3}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}} =$
 $\frac{3abcd}{bcd + acd + abd + abc} = 7 \frac{1797}{2289}$, for the number of days
 required.

QUESTION LXII.

To find that Number whose Square Root is to its Cube Root,
 in the Proportion of 5 to 2.

Let x^6 express the required number :
 then x^3 will be its square root,
 and x^2 its cube root.

Now $x^3 : x^2 :: 5 : 2$, by the question.

Therefore $2x^3 = 5x^2$, or $2x = 5$;

or, lastly, $x = \frac{5}{2} = 2.5$.

Whence $x^6 = 244.140625 =$ the number sought.

QUESTION LXIII.

To find two Numbers in the Proportion of 3 to 5; whereof
 the fifth Power of the first shall be to the third Power
 of the second, as 972 to 125.

If $3x$ be put for the first number,
 then $5x$ will express the second; and we shall
 have $(3x)^5 : (5x)^3 :: 972 : 125$, by the question,
 that is, $243x^5 : 125x^3 :: 972 : 125$.

Hence $243x^5 \times 125 = 125x^3 \times 972$,

or, $243x^2 = 972$.

Therefore $x^2 = \frac{972}{243} = 4$;

and $x = \sqrt{4} = 2$. So that 6 and 10 are the two
 numbers that were to be found.

QUESTION LXIV.

To find three Numbers in the Ratio of $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$; whereof the Sum of all the Squares shall be 549.

Let x denote the first number,

then it will be $\frac{1}{2} : \frac{1}{3} :: x : \frac{2x}{3}$ the second number.

And $\frac{1}{2} : \frac{1}{4} :: x : \frac{x}{2}$ the third number.

Hence $x^2 + \frac{2x}{3}^2 + \frac{x}{2}^2 = 549$, by the question.

that is, $x^2 + \frac{4x^2}{9} + \frac{x^2}{4} = 549$,

or, $36x^2 + 16x^2 + 9x^2 = 36 \times 549$.

From which $x^2 = \frac{36 \times 549}{61} = 36 \times 9$,

and $x = 6 \times 3 = 18$. Therefore 18, 12 and 9, are the three required numbers.

Otherwise.

By reducing the given fractions $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$, to the same denomination, they will appear to be in the proportion of 6, 4 and 3. If, therefore, the first of the three numbers sought be denoted by $6x$, the other two will be expressed by $4x$, and $3x$, respectively: and so we shall have

$$36x^2 + 16x^2 + 9x^2 = 549.$$

Whence $x = 3$, and the numbers sought, as before.

QUESTION LXV.

Having given the Difference of two Numbers = 6, and their Product = 720; to find the Numbers.

Let the lesser of them be denoted by x ; then the greater will be $x + 6$; and so, by the question, we shall have $xx + 6x = 720$.

But in order to the resolution of this equation (in which both the first and second powers of x are involved)

volved) let half the coefficient of x , which (in this case) is 3, be taken and squared, and let that square be added to both sides of the equation: by which means it becomes $xx+6x+9=729$.

Whereof the former part being now a compleat square, its root, may, therefore, be extracted; and will be expressed by the said half-coefficient joined to x with its proper sign; that is, by $x+3$ (as may be very easily proved by the multiplication of $x+3$ into itself; whence $xx+6x+9$, the very quantity above is produced.) Hence it is evident, that $x+3=\sqrt{729}=27$ (for equal quantities have equal square roots;) and consequently $x=24$.

QUESTION LXVI.

The Sum of two Numbers being given =60, and the Sum of their Squares =1872; to find the Numbers.

Let x be the greater of them;
then $60-x$ will be the lesser:

and therefore $x^2+60-x^2=1872$;

or $x^2+3600-120x+x^2=1872$;

whence $2x^2-120x=-1728$,

and $x^2-60x=-864$;

from which, by completing the square (as in the last problem) we get $x^2-60x+900=(-864+900)=36$;

and consequently, by taking the root, $x-30=\sqrt{36}=6$.

Therefore $x=6+30=36$: and $60-x=24$: which are the two numbers that were to found.

But, to solve the problem in a more general manner (by letters) put the sum of the two numbers $=a$, the sum of their squares $=b$, and the greater number $=x$, as before.

Then will $x^2+a^2-2ax+x^2=b$, by the question.

Hence $2x^2-2ax=b-a^2$,

and $x^2-ax=\frac{b}{2}-\frac{a^2}{2}$.

F 2

Where,

Where, half the coefficient of the second term being $\frac{a}{2}$, the completed square will therefore be

$$x^2 - ax + \frac{a^2}{4} \left(= \frac{b}{2} - \frac{a^2}{2} + \frac{a^2}{4} \right) = \frac{b}{2} - \frac{a^2}{4};$$

from which, by extracting the root,

$$\text{we have } x - \frac{a}{2} = \sqrt{\frac{b}{2} - \frac{aa}{4}};$$

and therefore $x = \sqrt{\frac{b}{2} - \frac{aa}{4}} + \frac{a}{2}$; which, if a be taken = 60, and $b = 1872$, will come out = 36, the very same as before.

QUESTION LXVII.

To divide the Number 60 (a) into two such Parts, that their Product may be 864 (b).

If x be put for the greater part, the lesser will be denoted by $a - x$; and we shall therefore have $ax - xx = b$; by the conditions of the question.

This equation, by changing the signs of all its terms (in order to have the highest power of x affirmative) becomes $xx - ax = -b$.

Whence, by completing the square,

$$\text{we have } xx - ax + \frac{aa}{4} = -b + \frac{aa}{4};$$

$$\text{and consequently } x - \frac{a}{2} = \sqrt{\frac{aa}{4} - b}.$$

$$\text{Therefore } x = \sqrt{\frac{aa}{4} - b} + \frac{a}{2} = 36, \text{ the greater part;}$$

and $a - x = 24$, the lesser.

QUESTION

QUESTION LXVIII.

To divide the Number 60 (a) into two Parts, so that the Square of the Greater multiplied by the Lesser, added to the Square of the Lesser multiplied by the Greater, may amount to 51840 (b).

If x be the greater part; then $a-x$ will be the lesser;
and $x^2 \times a-x + a-x^2 \times x = b$,
that is, $ax^2 - x^3 + a^2x - 2ax^2 + x^3 = b$,
or, $a^2x - ax^2 = b$.

Whence $x^2 - ax = -\frac{b}{a}$;

and, consequently, $x = \sqrt{\frac{a^2}{4} - \frac{b}{a}} + \frac{a}{2} = 36$.

QUESTION LXIX.

The Sum of two Numbers being given = 20 (a), and the Sum of their Cubes = 2240 (b); to determine the Numbers.

Let x be the greater; then $a-x$ will be the lesser;
and therefore $x^3 + a^3 - 3a^2x + 3ax^2 - x^3 = b$,
that is, $x^3 + a^3 - 3a^2x + 3ax^2 - x^3 = b$:
whence $3ax^2 - 3a^2x = b - a^3$,
and $x^2 - ax = \frac{b}{3a} - \frac{a^2}{3}$.

Therefore, by completing the square;

$$x^2 - ax + \frac{aa}{4} \left(= \frac{b}{3a} - \frac{a^2}{3} + \frac{a^2}{4} \right) = \frac{b}{3a} - \frac{a^2}{12}:$$

and, by extracting the root, $x - \frac{a}{2} = \sqrt{\frac{b}{3a} - \frac{aa}{12}}$.

Consequently $x = \frac{a}{2} + \sqrt{\frac{b}{3a} - \frac{aa}{12}} = \frac{20}{2} + \sqrt{37\frac{1}{2} - 33\frac{1}{2}} = 12$.

QUESTION

QUESTION LXX.

To divide the Number 240 (a) into two such Parts; that the greater Part divided by the Lesser, may be to the lesser Part divided by the Greater, in the Proportion of 147 to 75 (or of m to n).

If the greater part be denoted by x , the lesser will be expressed by $a-x$; and we shall have

$$\frac{x}{a-x} : \frac{a-x}{x} :: m : n.$$

$$\text{Hence } \frac{nx}{a-x} = \frac{m \times a-x}{x}$$

$$\text{and } \frac{nx^2}{m} = (a-x)^2;$$

from which, by extracting the square root, on both sides, $x\sqrt{\frac{n}{m}} = a-x.$

$$\text{Whence, putting } \sqrt{\frac{n}{m}} = \frac{b}{c},$$

$$\text{we have } bx = ca - cx; \text{ and consequently } x = \frac{ca}{b+c}.$$

$$\text{But, in the case above proposed, } \sqrt{\frac{n}{m}} \text{ being } =$$

$$\sqrt{\frac{75}{147}} = \sqrt{\frac{25}{49}} = \frac{5}{7}, \text{ we have } b=5, c=7; \text{ and}$$

$$\text{therefore } x = \frac{7 \times 240}{12} = 7 \times 20 = 140.$$

QUESTION

QUESTION LXXI.

Two Workmen, A and B, were employed, by the Day, at different Rates: A at the end of a certain Number of Days, had 96 Shillings to receive; but B, who played 6 of those Days, received only 54 Shillings: but, had B worked the whole Time, and A played 6 Days, they would have received exactly alike. It is proposed to find the Number of Days they were employed; and what each had a Day.

Let x be the number of days that A worked; then $x-6$ will be the days that B worked.

Moreover $\frac{96}{x}$, will be the wages of A per day;

and $\frac{54}{x-6}$, the wages of B per day.

Therefore $\frac{54}{x-6} \times x$, is what B would have earned, had he worked the whole time:

and $\frac{96}{x} \times x-6$, what A would have earned had he played 6 days. Which two values being equal, by the question, we have $\frac{54x}{x-6} = \frac{96 \times x-6}{x}$.

Whence, by reduction, $54x^2 = 96 \times x-6^2$,

or, $\frac{9x^2}{16} = x-6^2$: therefore, by extracting the square root on both sides, $\frac{3x}{4} = x-6$; and consequently $x=24$.

From which it is evident, that A had 4 shillings, and B 3 shillings, a day.

QUESTION

QUESTION LXXII.

From two Places, at the Distance of 320 (a) Miles, two Persons, A and B, set out, at the same Time, in order to meet each other; A travelled b (b) Miles a Day more than B; and the Number of Days in which they met was equal to Half the Number of Miles B went in a Day. It is required to find how far each travelled to meet the other.

Let x be the number of days in which they met; then, by the $\left\{ \begin{smallmatrix} 2x \\ 2x+b \end{smallmatrix} \right\}$ question, $\left\{ \begin{smallmatrix} B \\ A \end{smallmatrix} \right\}$ will be the number of miles $\left\{ \begin{smallmatrix} B \\ A \end{smallmatrix} \right\}$ went a day.

Therefore, by multiplying each of these by (x) the number of days, we have $2xx$, and $2xx+bx$, for the whole number of miles travelled by B and A, respectively:

and consequently $4xx+bx=a$.

$$\text{Hence } xx + \frac{bx}{4} + \frac{bb}{64} = \frac{a}{4} + \frac{bb}{64},$$

$$\text{and } x = \sqrt{\frac{a}{4} + \frac{bb}{64} - \frac{b}{8}} = 8.$$

Therefore $2xx=128$ = the miles travelled by B; and $2xx+bx=192$ = those travelled by A.

QUESTION LXXIII.

Two Messengers, A and B, were dispatched at the same Time, to a Place, at the Distance of 90 (a) Miles; the former of whom, by riding one Mile an Hour more than the other, arrived at the end of his Journey one Hour before him. The Question is, to find at what Rate each travelled per Hour.

If x be the miles that A rode per hour; then $x-1$ will be the miles which B rode per hour: moreover $\frac{a}{x}$ will be the number of hours in which A performed

performed the whole journey; and $\frac{a}{x-1}$ will be the number of hours wherein B performed it.

And therefore $\frac{a}{x} = \frac{a}{x-1} - 1$, by the question.

whence, by reduction $ax - a = ax - x^2 + x$,
or $x^2 - x = a$.

Therefore $x^2 - x + \frac{1}{4} = a + \frac{1}{4}$ (by completing the square)
and consequently $x = \sqrt{a + \frac{1}{4} + \frac{1}{4}} = 9\frac{1}{2} + \frac{1}{2} = 10$.

QUESTION LXXIV.

To find two Numbers, so that their Sum multiplied by the Greater, may produce 100 times the Lesser, and being multiplied by the Lesser may produce 64 times the Greater.

Let x denote the greater number, and y the lesser.

Then $x + y \times x = 100y$,

And $x + y \times y = 64x$.

Now, the first of these equations being multiplied by y , and the second by x , they become both alike;

and so we have $100y^2 = 64x^2$:

therefore, by taking the square root, $10y = 8x$,

and consequently $y = \frac{4x}{5}$: which value substituted in

the second equation, gives $x + \frac{4x}{5} \times \frac{4x}{5} = 64x$;

or, $\frac{9x}{5} \times \frac{4}{5} = 64$.

Whence $x = \frac{25 \times 16}{9} = 44\frac{4}{9}$; and $y = 35\frac{5}{9}$.

QUESTION .LXXV.

To find three Numbers, so that their continual Product divided by the Sum of each two of them, may quote given Numbers; or (which is the same Thing) to determine the Values of x , y and z , in the underwritten Equations.

$$\frac{xyz}{x+y}=200, \frac{xyz}{x+z}=150, \frac{xyz}{y+z}=120.$$

First, by multiplication,

$$xyz=200x+200y=150x+150z=120y+120z:$$

therefore, by reduction,

$$\left. \begin{array}{l} 50x+200y=150z \\ 200x+80y=120z \end{array} \right\} \text{ or } \left\{ \begin{array}{l} x+4y=3z, \\ 5x+2y=3z. \end{array} \right.$$

Hence $x+4y=5x+2y$, and therefore $y=2x$:

which, substituted in $3z=x+4y$, gives $3z=9x$; and $z=3x$.

And, by substituting for both y and z , in the equation

$$\frac{xyz}{x+y}=200, \text{ we get } \frac{x \times 2x \times 3x}{x+2x}=200;$$

that is, $2x^2=200$.

Consequently $x=10$, $y=20$, and $z=30$.

QUESTION .LXXVI.

To find the Ratio of two Numbers, whose Rectangle is equal to the Square of their Difference.

Let the lesser number be to the greater as 1 is to x ; then, if the said lesser number be denoted by z , the greater will be expressed by xz , and we shall have $xz \times z = [xz - z]^2$, or $xz^2 = x^2z^2 - 2xz^2 + z^2$ (by the question). Whence, dividing the whole by z^2 , there results $x = x^2 - 2x + 1$.

Therefore $x^2 - 3x = -1$;

and consequently $x = \frac{3 + \sqrt{5}}{2} = 2.618 \text{ \&c.}$

Hence

Hence the ratio of any two numbers, whose rectangle is equal to the square of their difference, must be that of 2.618 &c. to unity.

QUESTION LXXVII.

To find two Numbers, whose Product is 300 (a); so that, if 10 (b) be added to the Lesser, and 8 (c) subtracted from the Greater, the Product of the Sum and Remainder shall, also, be equal to 300 (a).

Let the greater number be denoted by x , and the lesser by y ;

then will $\left\{ \begin{array}{l} xy = a \\ x - c \times y + b = a \end{array} \right\}$ by the problem.

By the last of which $xy + bx - cy - cb = a$.

From whence, the first equation being subtracted, there rests $bx - cy - cb = 0$:

therefore $bx = cb + cy$, and $x = \frac{cb + cy}{b}$: which substituted

in the first equation, gives $\frac{cby + cy^2}{b} = a$.

From which we have $y^2 + by = \frac{ab}{c}$:

and, by completing the square, $y^2 + by + \frac{bb}{4} = \frac{ab}{c} + \frac{bb}{4}$.

Whence $y = \sqrt{\frac{ab}{c} + \frac{bb}{4}} - \frac{b}{2} = 15$;

and $x \left(= \frac{a}{y} \right) = 20$.

QUESTION LXXVIII.

To divide 100 into two such Parts, that their Difference may be to their Sum, as their Rectangle to the Difference of their Squares.

Let a represent half the given number, and x half the difference of its two parts; then, the greater of them

them being expressed by $a+x$, and the lesser by $a-x$,
 it will be $2x : 2a :: a+x \times a-x : a+x^2 - a-x^2$,
 that is, by reduction, $x : a :: aa-xx : 4ax$;
 hence $4ax^2 = a \times aa - xx$,
 or $4x^2 = a^2 - x^2$,

and therefore $x = \sqrt{\frac{aa}{5}} = 22,36$.

So that, the greater part is 72,36; and the lesser 27,64.

QUESTION LXXIX.

To divide the Number 60 (a) into two such Parts, that
 their Product may be to the Sum of their Squares, as
 the Ratio of 2 (m) to 5 (n).

Let x be the greater part;
 then, the lesser part will be $a-x$,
 the product $ax-xx$,
 and the sum of the squares $a^2-2ax+2x^2$.
 Therefore $m : n :: ax-xx : a^2-2ax+2x^2$;
 and so, $2mx^2-2max+ma^2=nax-nx^2$.
 Whence, $2mx^2+nx^2-2max-nax=-ma^2$;
 that is, $2m+n \times x^2-2m+n \times ax=-ma^2$;

or, $x^2-ax = -\frac{maa}{2m+n}$.

Therefore $x^2-ax+\frac{aa}{4} = \left(\frac{aa}{4} - \frac{ma^2}{2m+n}\right) = \frac{aa}{4} \times \frac{n-2m}{n+2m}$,

and $x = \frac{a}{2} \sqrt{\frac{n-2m}{n+2m}} + \frac{a}{2} = 40$.

QUESTION LXXX.

To find two Numbers whose Product shall be 320 (a), and
 the Difference of their Cubes to the Cube of their Dif-
 ference, as 61 (n) is to Unity.

Let x be the greater number, and y the lesser;
 then will $xy=a$,
 and $x^3-y^3 : x-y)^3 :: n : 1$, by the question.

Which

Which, by actually involving $x-y$, becomes $x^3-y^3 : x^3-3x^2y+3xy^2-y^3 :: n : 1$.
From whence (by subtracting the consequents from their antecedents)

we get $3x^2y-3xy^2 : x^3-3x^2y+3xy^2-y^3 :: n-1 : 1$;
or, which is the same, $3xy \times x-y : x-y)^3 :: n-1 : 1$.
Whence, dividing by $x-y$,

we have $3xy : x-y)^3 :: n-1 : 1$;

or, $3a : x-y)^3 :: n-1 : 1$ (because $xy=a$).

From whence $x-y)^3 = \frac{3a}{n-1}$;

and consequently $x-y = \sqrt[3]{\frac{3a}{n-1}}$; which put $=b$.

Then from the equations $xy=a$, and $x-y=b$, the value of x will be found $= \frac{\sqrt{b^2+4a}+b}{2}$; and

that of $y = \frac{\sqrt{b^2+4a}-b}{2} = 16$, *vid. Prob. 65.*

The same otherwise.

Let z denote the half sum, and x the half difference, of the two numbers; then the greater will be expressed by $z+x$, and the lesser by $z-x$; and we shall therefore

have $\left\{ \begin{array}{l} z+x \times z-x = a \\ (z+x)^3 - (z-x)^3 = n \times 2x^3 \end{array} \right\}$ *by the question.*

that is $\left\{ \begin{array}{l} z^2 - x^2 = a, \\ 6zx + 2x^3 = 8nx^3. \end{array} \right.$

The last of which, divided by $2x$,

gives $3z^2 + x^2 = 4nx^2$.

From whence, the treble of the first being deducted, we have $4x^2 = 4nx^2 - 3a$; and consequently $x =$

$$\sqrt{\frac{3a}{4n-4}} = 2.$$

Hence $z (= \sqrt{a+xx}) = 18$; therefore $z+x=20$,
and

and $x-x=16$; which are the two numbers that were to be found,

QUESTION LXXXI.

A Farmer received 7l. 4s. for a certain Quantity of Wheat, and an equal Sum, at a Price less by 1s. 6d. a Bushel, for a Quantity of Barley, which exceeded that of the Wheat by 16 Bushels. How many Bushels were there of each?

Put a = the total value of each sort of grain,

b = the difference of the quantities,

c = the difference of the prices per bushel,

and x = the number of bushels of the wheat.

Then, dividing the whole price by the number of bushels, we have $\frac{a}{x}$ for the price of the wheat per bushel: and, in the same manner, the price of the barley per bushel will appear to be $\frac{a}{x+b}$.

Therefore $\frac{a}{x} - \frac{a}{x+b} = c$, by the question.

Whence (by reduction) $ax + ab - ax = cx^2 + bcx$;

or, $\frac{ab}{c} = x^2 + bx$.

Consequently $x = \sqrt{\frac{ab}{c} + \frac{bb}{4}} - \frac{b}{2} = 32$, the number of bushels of wheat; and $x + 16 = 48$, the number of bushels of barley.

QUESTION

QUESTION LXXXII.

One bought two Pieces of Cloth of different Sorts, whereof the finer cost 4 Shillings a Yard more than the other; so that, for the finest, he paid 360 Shillings; whereas the coarsest, which exceeded the finest by 10 Yards, cost him, only, 320 Shillings. How many Yards were there of each Piece?

Let x be the number of yards of the finest, and y , the number of shillings a yard: then $x+10$, will be the length of the coarsest piece, and $y-4$, its price per yard.

Hence $\left\{ \begin{array}{l} xy = 360 \\ x+10 \times y-4 = 320 \end{array} \right\}$ by the question.

By the last equation $xy+10y-4x=360$; from whence deducting $xy=360$, we have $10y-4x=0$.

Therefore $10y=4x$; and $y=\frac{2x}{5}$. Which, being sub-

stituted in the first equation, we get $\frac{2x^2}{5} = 360$.

Whence x comes out $=\sqrt{900}=30$.

QUESTION LXXXIII.

There are two Numbers, whose Rectangle is equal to the Difference of their Squares; and the Sum of their Squares is also equal to the Difference of their Cubes. What are those Numbers?

Let x denote the lesser number, and let the greater be in proportion thereto as y is to unity; or, which is the same thing, let the greater number be denoted by xy .

Then $\left\{ \begin{array}{l} x \times xy = x^2 y^2 - x^2 \\ x^2 y^2 + x^2 = x^3 y^3 - x^3 \end{array} \right\}$ by the problem.

Therefore $\left\{ \begin{array}{l} y = y^3 - 1 \\ y^2 + 1 = xy^3 - x \end{array} \right\}$ by division.

From

From the first of which equations, $y^2 - y = 1$;

whence $y^2 - y + \frac{1}{4} = \frac{5}{4}$, and consequently $y = \frac{1 + \sqrt{5}}{2}$.

But, by the second equation, $x = \frac{yy+1}{yy-1} = \frac{y+2}{2y}$ (be-

cause $yy = y + 1$) $= \frac{\frac{1}{2} + \frac{1}{y}}{\frac{1}{2} + \frac{1}{y}} = \frac{1}{2} + \frac{2}{\sqrt{5}+1} = \frac{1}{2} + \frac{2\sqrt{5}-2}{4}$

(by multiplying both the numerator and denominator by $\sqrt{5}-1$) $= \frac{\sqrt{5}}{2}$. Therefore the two quantities

sought are $\frac{1}{2}\sqrt{5}$, and $\frac{5+\sqrt{5}}{4}$.

QUESTION LXXXIV.

A sets out from London for York, at the same Time as B sets out from York for London; and the Rate at which they travel is such, that A, 9 Hours after their meeting arrives at York, and B at London, in 16 Hours after. The Question is, to find in what Time each Traveller performs his Journey.

Let x denote the number of hours travelled by each before the time of their meeting on the road.

Then, since A (by the former part of the question) goes over the very same ground in 9 (*a*) hours, as B travelled in x hours, we have, therefore, as $a : x :: x :$

$\frac{xx}{a}$, the time wherein B travels a distance equal to that gone over by A, in x hours: but (by the latter part of the question) B, in 16 (*b*) hours, travels the very same distance as A in x hours. Hence it is evident that $\frac{xx}{a}$ and b are equal to each other:

and consequently that $x = \sqrt{ab} = 12$.

Therefore

Therefore A performs the journey in 21 hours, and B in 28 hours.

QUESTION LXXXV.

The Sum of 190l. was divided between three Persons; whose Shares were in Geometrical Proportion, and the greatest of them exceeded the least by 50l. What were all the several Shares?

If x be put for the least of them, then the greatest will be $x + 50$; the sum of which two, subtracted from (190) the whole, leaves $140 - 2x$ for the mean share.

Therefore, $x : 140 - 2x :: 140 - 2x : x + 50$, by the question.

And consequently $(140 - 2x)^2 = x \times x + 50$; that is, $19600 - 560x + 4x^2 = x^2 + 50x$.

Whence $3x^2 - 610x = -19600$,

and $x^2 - \frac{610x}{3} = -\frac{19600}{3}$.

Hence, by completing the square,

$$x^2 - \frac{610x}{3} + \frac{93025}{9} \left(= -\frac{19600}{3} + \frac{93025}{9} \right) = \frac{34225}{9};$$

and, by extracting the root, $x - \frac{305}{3} = \pm \frac{185}{3}$.

Therefore $x = \frac{305 - 185}{3} = 40$: and so the other two shares are 60 and 90 pounds.

QUESTION LXXXVI.

Two Notes, one of 120l. payable in 6 Months, and the other of 150l. payable in 9 Months, were discounted for 8l. 10s. What Rate of Interest were they discounted at?

Let x denote the interest of one pound for 12 months. Then the amount of 1l. in 6 months being $1 + \frac{x}{2}$, and,

H

in

in 9 months $1 + \frac{3x}{4}$ the *present value* of the bill, due

at the end of 6 months, will therefore be $\frac{120}{1 + \frac{1}{2}x}$; and

that of the bill, due at the end of 9 months, $\frac{150}{1 + \frac{3}{4}x}$;

Whence, we have $\frac{120}{1 + \frac{1}{2}x} + \frac{150}{1 + \frac{3}{4}x} (=120 + 150 - 8.5)$
 $= 261.5$ (by the question.)

Which, by reduction, becomes $120 + 90x + 150 + 75x$
 $= 261.5 \times 1 + \frac{1}{4}x + \frac{3}{8}x^2$;

or, $270 + 165x = \frac{261.5 \times 8 + 10x + 3x^2}{8}$.

Therefore $\frac{270 + 165x \times 8}{261.5} = 3x^2 + 10x + 8$,

that is, $\frac{2160}{261.5} + \frac{1320x}{261.5} = 3x^2 + 10x + 8$,

From which we have

$$x^2 + \frac{2590}{3 \times 523} \times x = \frac{136}{3 \times 523}$$

Whence, by completing the square, &c.

$$x \text{ is found} = \sqrt{\frac{136}{1569} + \frac{1295}{1569}}^2 - \frac{1295}{1569} =$$

$$\frac{\sqrt{136 \times 1569 + 1295 \times 1295 - 1295}}{1569} = 0.05093. \text{ Which}$$

multiplied by 100, gives 5.093*l.* or 5*l.* 1*s.* 10½*d.* nearly, for the rate *per cent.* at which the notes were discounted.

QUESTION

QUESTION LXXXVII.

A and B take, in Trade, 5940l. per Annum, each; but A, whose Profits are 2 per Cent. greater than those of B, clears 100l. per Annum more than B. What are the Profits of each, per Cent? And what do they clear per Annum?

Let $c=100$ (pounds),

$b (=5940)$ =the whole sum taken by each,

$d (=100l.)$ =the given difference of their gains,

x =what A gains per cent.

$x-a (=x-2)$ =what B gains per cent.

Now, since A, in taking $c+x$ pounds, gains x pounds,

it will be $c+x : x :: b : \frac{bx}{c+x}$, the whole gain of A :

and, in the same manner, we have,

$c+x-a : x-a :: b : \frac{b \times x-a}{c+x-a}$, the whole gain of B :

therefore $\frac{bx}{c+x} - \frac{b \times x-a}{c+x-a} = d$, by the question.

Hence $abc = x^2 + 2cx - ax - ac + cc \times d$;

and consequently $x^2 + 2c - a \times x = \frac{abc}{d} + ac - cc$.

Whence, by completing the square;

$$x^2 + 2c - a \times x + \frac{(2c-a)^2}{4} = \frac{abc}{d} + \frac{aa}{4};$$

$$\text{and therefore } x = \sqrt{\frac{abc}{d} + \frac{aa}{4} - \frac{2c-a}{2}} = 10.$$

From which it appears that A gained 10 per cent. and cleared 540l. per annum; and that B gained 8 per cent. and cleared 440l. per annum.

QUESTION LXXXVIII.

Of four Numbers in Geometrical Progression, there is given the Sum of the two least = 20 (a) and the Sum of the two greatest = 45 (b); to find the Numbers.

Let x denote the first number, and y the third; then the second being expressed by $a-x$, and the fourth by $b-y$, the four terms of the progression, placed in order, will stand thus, x , $a-x$, y , $b-y$.

Whence, by the nature of proportionals,

$$\text{we have } \begin{cases} x \times b - y = a - x \times y, \\ xy = a - x^2. \end{cases}$$

From the first of which equations y is $= \frac{bx}{a}$: and, by

substituting in the second, we have $\frac{bx^2}{a} = a - x^2$.

Whence, by extracting the square root on both sides,

$$x \sqrt{\frac{b}{a}} = a - x;$$

$$\text{and consequently } x = \frac{a}{1 + \sqrt{\frac{b}{a}}} = 8.$$

Therefore 8, 12, 18, and 27, are the four numbers that were to be found.

QUESTION LXXXIX.

The Sum of 700l. (a) was divided among four Persons, A, B, C and D; whose Shares were in Geometrical Progression; and the Difference between the greatest and least, was to the Difference between the two Means, as 37 (m) to 12 (n). What were all the several Shares?

Let the share of A, or the first term of the progression, be denoted by x , and let the common ratio be that of 1 to y .

Then

Then $x+xy+xy^2+xy^3=a$ } by the question.
and $xy^3-x:xy^2-xy::m:n$

From which proportion we have

$y^3-1=y-1 \times \frac{my}{n}$, or $y^2+y+1=\frac{my}{n}$ (by dividing the whole by $y-1$).

Hence y is found $=\frac{m-n+\sqrt{mm-2mn-3nn}}{2n}=\frac{25+7}{24}$

$=\frac{4}{3}$. Whence $x (= \frac{a}{1+y+y^2+y^3})$ is given $=$

$\frac{27 \times 700}{27+36+48+64}=108$. Therefore the four shares are 108, 144, 192, and 256*l*.

QUESTION XC.

Of four Numbers in Arithmetical Progression, the Sum of the Squares of the two Means is given =400 (a); and the Sum of the Squares of the two Extremes =464 (b). To determine the Numbers.

If x be put for the lesser extreme, and y for the common difference; then the four numbers will be expressed by,

$x, x+y, x+2y$, and $x+3y$ respectively; and we shall therefore have $\left\{ \begin{array}{l} x+y \\ x^2+x+3y \end{array} \right\}^2 = a$ } or

$\left\{ \begin{array}{l} 2x^2+6xy+5y^2=a \\ 2x^2+6xy+9y^2=b \end{array} \right\}$ by the question.

Whence, by subtraction, $4y^2=b-a$:

and therefore $y=\frac{\sqrt{b-a}}{2}=4$.

But, by the first equation, $x^2+3xy=\frac{a}{2}-\frac{5yy}{2}$:

from which (looking upon y as known) we get

$x=\sqrt{\frac{a}{2}-\frac{yy}{4}-\frac{3y}{2}}=8$.

Therefore

Therefore 8, 12, 16, and 20, are the four numbers that were to be found,

QUESTION XCI.

The Sum of four Numbers, in Arithmetical Progression; being given = 56 (b), and the Sum of their Squares = 864 (c); to find the Numbers.

If half the sum of the two middle numbers be denoted by a , and half their difference by x , the numbers themselves will be expressed by $a-x$, and $a+x$: and we shall have

$$\left\{ \begin{array}{l} a-3x+a-x+a+x+a+3x=b \\ (a-3x)^2+(a-x)^2+(a+x)^2+(a+3x)^2=c \end{array} \right\} \text{by the nature of the question.}$$

Hence, by reduction, $4a=b$, and $4aa+20xx=c$.

$$\text{Therefore } a = \frac{b}{4} = 14; \text{ and } x = \sqrt{\frac{c}{20} - \frac{aa}{5}} = 2.$$

From which the numbers themselves are given; being 8, 12, 16 and 20.

By the same way of proceeding the problem may be resolved, when the progression is supposed to consist of 6, 8, 10, or any other, even, number (n) of terms.

For the sum of the squares of the two means ($a-x$, and $a+x$) being $= 2 \times aa + xx$; and the sum of the squares of the two terms ($a-3x$ and $a+3x$) next adjacent to them $= 2 \times aa + 9xx$; also the sum of the squares of the two terms ($a-5x$ and $a+5x$) next adjacent to these last being $= 2 \times aa + 25xx$, &c. &c. it follows that

$$2 \times aa + x + 2 \times aa + 9xx + 2 \times aa + 25xx + \&c. = c$$

Which equation, by putting $1+9+25+36+ \&c.$

(continued to $\frac{n}{2}$ terms) $= f$, is reduced to $naa + 2fxx$

$= c.$

Whence

Whence $x = \sqrt{\frac{c - naa}{2f}}$; from which, as a is given $= \frac{b}{n}$, the value of x will also be known.

QUESTION XCII.

Having the Sum (b) and the Sum of the Squares (c) of five Numbers, in Arithmetical Progression; to determine the Progression,

Let a denote the middle number, and x the common difference; then the five terms of the progression will be, $a - 2x$, $a - x$, a , $a + x$, and $a + 2x$; whence we have $5a = b$, and $5aa + 10xx = c$, by the conditions of the question. From which a is found $= \frac{b}{5}$, and $x (=$

$$\sqrt{\frac{c}{10} - \frac{aa}{2}} = \sqrt{\frac{c}{10} - \frac{bb}{50}}.$$

After the same manner the problem may be investigated, when any odd number (n) of terms is propounded: for the sum of the squares of the 2 terms ($a - x$ and $a + x$) adjacent to the middle one being $= 2 \times aa + xx$; and that of the two terms ($a - 2x$ and $a + 2x$) next to these $= 2 \times aa + 4xx$, &c. &c. it is evident therefore that

$aa + 2 \times aa + xx + 2 \times aa + 4xx + 2 \times aa + 9xx + \&c. = c$.
Which equation, by putting $1 + 4 + 9 + 16 + \&c.$
(to $\frac{n-1}{2}$ terms) $= f$, becomes $naa + 2fxx = c$.

Hence $x = \sqrt{\frac{c - naa}{2f}}$; from which, as a is given $= \frac{b}{n}$, all the terms of the progression will be known.

By a like process, if, instead of the sum of the squares, the sum of the cubes, or biquadrates, be given, the problem may be resolved,

QUESTION

QUESTION XCIII.

A Traveller, bound to a certain Place at the Distance of 140 Miles, goes 26 Miles the first Day, 24 Miles the second Day, 22 Miles the third; and so on, in Arithmetical Progression, decreasing 2 Miles every Day. In how many Days will he arrive at the End of his Journey?

Put $b=140$, the whole, given, distance,

$c=26$, the part thereof travelled in the first day,

$d=2$, the common difference by which each day's journey decreases,

and x =the number of days wherein the whole journey is performed.

Then, from the nature of the question, it is evident, that, the last day's journey will fall short of the first, by $x-1$ times the common difference (d); and is, therefore, truly expressed by $c-x-1 \times d$.

But it is well known that the sum of any arithmetical progression is equal to the sum of the first and last terms multiplied by half the number of the terms:

hence we have $c+c-x-1 \times d \times \frac{x}{2}$ for the whole distance travelled; and consequently this equation,

$$2c-x-1 \times d \times \frac{x}{2}=b.$$

Whence $2cx-dx^2+dx=2b$,

$$\text{and } xx-\frac{2c+d}{d} \times x = -\frac{2b}{d}.$$

$$\text{From which } x = +\sqrt{\frac{(2c+d)^2}{4dd}-\frac{2b}{d}}+\frac{2c+d}{2d}=7.$$

QUESTION

QUESTION XCIV.

After A, who travelled at the Rate of 4 Miles an Hour, had been set out two Hours and three Quarters, B set out to overtake him; and in order thereto went $4\frac{1}{2}$ Miles the first Hour, $4\frac{1}{2}$ the second, 5 the third; and so on, gaining a Quarter of a Mile every Hour. How many Hours did it take him to come up with A?

Put $a=4$, the distance travelled by A, per hour.

$b=11$, the distance gone by A before B set out,

$c=4\frac{1}{2}$, the miles travelled by B in the first hour,

$d=\frac{1}{4}$ of a mile, the common excess,

and x =the number of hours required.

Then it is evident (from the preceding problem) that the distance travelled by B, in the last hour, will be $c+x-1 \times d$; and that $2c+x-1 \times d \times \frac{x}{2}$ will express the

whole distance travelled by B in x hours.

But the distance of A in x hours being ax , the whole distance travelled by A will therefore be $ax+b$; which being equal to that of B (by the question) we thence have,

$$2c+x-1 \times d \times \frac{x}{2} = ax+b;$$

and consequently $2cx+dx^2-dx=2ax+2b$;

$$\text{or } xx + \frac{2c-2a-d}{d} \times x = \frac{2b}{d}.$$

Which (by making $\frac{2c-2a-d}{d}=f$) gives

$$x = \sqrt{\frac{2b}{d} + \frac{ff}{4}} - \frac{f}{2} = 8, \text{ the number of hours required.}$$

QUESTION XCV.

To find four Numbers in Arithmetical Progression, which being increased by 2, 4, 8 and 15, respectively, the Sums shall be in Geometrical Progression.

Let x denote the least number, and y the common difference. Then the four numbers will be expressed by x , $x+y$, $x+2y$, and $x+3y$;

and the four specified sums,

by $x+2$, $x+y+4$, $x+2y+8$, and $x+3y+15$.

Whence, by the nature of geometrical proportionals,

we have $\begin{cases} x+2 \times x+2y+8 = x+y+4^2, \\ x+2 \times x+3y+15 = x+y+4 \times x+2y+8, \end{cases}$

that is $\begin{cases} y^2+4y=2x \\ 2y^2+10y+2=5x \end{cases}$ } by multiplication, and transposition.

Hence $5y^2+20y=4y^2+20y+4$;

therefore $y^2=4$, and $y=2$;

From which $x=6$; so that the other three numbers are 8, 10, and 12 respectively.

For 6, 8, 10 and 12 are in arithmetical progression; and $6+2$, $8+4$, $10+8$, and $12+15$, are in geometrical progression.

QUESTION XCVI.

The Rectangle (a) and the Sum of the Cubes (b) of two Numbers being given; to determine the Numbers.

Let x denote the greater, and y the lesser number:

then will $\begin{cases} xy=a \\ x^3+y^3=b \end{cases}$ } by the question.

Whence, by involution, $x^3y^3=a^3$,

and $x^6+2x^3y^3+y^6=b^2$.

Let the quadruple of the former of these two equations be subtracted from the latter, and you will have $x^6-2x^3y^3+y^6=b^2-4a^3$;

and,

and, by extracting the square root, on both sides,
 $x^3 - y^3 = \sqrt{b^2 - 4a^3}$. Which, added to the first equation
 gives $2x^3 = b + \sqrt{b^2 - 4a^3}$.

Consequently $x = \sqrt[3]{\frac{1}{2}b + \frac{1}{2}\sqrt{bb - 4a^3}}$;

and $y\left(=\frac{a}{x}\right) = \sqrt[3]{\frac{1}{2}b + \frac{1}{2}\sqrt{bb - 4a^3}}$.

The same otherwise.

Since $xy = a$, we have $y = \frac{a}{x}$; and therefore

$x^3 + \frac{a^3}{x^3} = b$, by the question.

Whence $x^6 + a^3 = bx^3$,

or, $x^6 - bx^3 = -a^3$.

Therefore, by completing the square,

$$x^6 - bx^3 + \frac{bb}{4} \left(= \frac{bb}{4} - a^3 \right) = \frac{bb - 4a^3}{4};$$

and, by extracting the root, $x^3 - \frac{b}{2} = \frac{\sqrt{bb - 4a^3}}{2}$.

Hence $x^3 = \frac{1}{2}b + \frac{1}{2}\sqrt{bb - 4a^3}$;

and consequently the values of x and y , *the very same as before.*

From either of the above solutions, a general theorem, for the resolution of cubic equations (according to the manner of *Cardan*) is very easily deducible.

For, by putting $z = x + y$, and involving each side to the third power, we have $z^3 = x^3 + 3x^2y + 3xy^2 + y^3 = x^3 + y^3 + 3xy(x + y) = x^3 + y^3 + 3a \times z$ (by substituting for xy and $x + y$, their equals, a and z). From whence, by transposition,

$$z^3 - 3az (=x^3 + y^3) = b.$$

But it appears from above, that $z (=x + y)$

$$\text{is} = \sqrt[3]{\frac{1}{2}b + \frac{1}{2}\sqrt{bb-4a^3}} + \sqrt[3]{\frac{1}{2}b + \frac{1}{2}\sqrt{bb-4a^3}};$$

Which value, therefore, is the true root of the equation $z^3 - 3az = b$.

From which the root of the equation $z^3 + cz = b$ (where the second term is positive) will also be given, by assuming $-3a = c$, and substituting for a : whence z is found

$$= \sqrt[3]{\frac{b}{2} + \sqrt{\frac{bb}{4} + \frac{c}{3}}} - \sqrt[3]{\frac{b}{2} + \sqrt{\frac{bb}{4} + \frac{c}{3}}}.$$

QUESTION XCVII.

The Difference of two Numbers being given = 4, and the Sum of their Cubes = 2240; to determine the Numbers.

Let x denote their half sum, and d their half difference, then the greater being $x+d$, and the lesser $x-d$, we have

$$(x+d)^3 + (x-d)^3 = 2240,$$

$$\text{that is } 2x^3 + 6d^2x = 2240,$$

$$\text{or, } x^3 + 3d^2x = \frac{2240}{2}.$$

$$\text{Put } c = 3d^2 (=12), \text{ and } b = \frac{2240}{2} (=1120)$$

then it will become $x^3 + cx = b$:

from whence by the general theorem, in the last problem, $x =$

$$\sqrt[3]{\frac{b}{2} + \sqrt{\frac{bb}{4} + \frac{c}{3}}} - \sqrt[3]{\frac{b}{2} + \sqrt{\frac{bb}{4} + \frac{c}{3}}} = 10.$$

Therefore 12 ($=x+d$) and 8 ($=x-d$) are the two numbers that were to be found.

QUESTION

QUESTION XCVIII.

It is proposed to divide the Number 24 (2a) into two such Parts, that the Difference of their Cubes may be 3584 (2b).

Let $a+x$ express the greater part, and $a-x$ the lesser; then will $(a+x)^3 - (a-x)^3 = 2b$; that is, $6a^2x + 2x^3 = 2b$; or $x^3 + 3a^2x = b$.

Put $c = 3a^2$, and $\sqrt[3]{\frac{b}{2} + \sqrt{\frac{bb}{4} + \frac{c^3}{3}}} = r$.

Then will $x^3 + cx = b$,

and $x = r - \frac{\frac{1}{3}c}{r} = 4$, by the theorem in the preceding page.

Whence 8 and 16 are the two required numbers.

QUESTION XCIX.

The Sum of the Squares of two Numbers being given = 208 (g) and the Sum of their Cubes = 2240 (h); to find the Numbers.

Let the greater number be denoted by $x+y$, and the lesser by $x-y$:

then will $\begin{cases} 2x^2 + 2y^2 = g \\ 2x^3 + 6xy^2 = h \end{cases}$ by the question.

From the first of these equations, multiplied by $3x$, let the second be subtracted;

whence you will have $4x^3 = 3gx - h$;

and consequently $x^3 - \frac{3gx}{4} + \frac{h}{4} = 0$;

that is, in numbers, $x^3 - 156x + 560 = 0$;

the roots of which equation (by Sect. 12 of my Treatise of Algebra) will be found to be 10, 4, and -14; whereof the first, only, is for our purpose: by means

of which we get $y = (\sqrt{\frac{1}{2}g - x^2}) = 2$.

Therefore

Therefore 12 and 8 are the two numbers that fulfil the conditions of the problem.

Note. The resolution of the above equation, by the theorem referred to in the preceding examples, is impossible; because the square root of a negative quantity is to be extracted; as, upon trial, will be found.

QUESTION C.

The Sum (a) and the continual Product (b) of four Numbers, in Geometrical Progression, being given; to determine the Numbers.

If the lesser of the two means be represented by $x-y$, and the greater by $x+y$, we shall have

$$\frac{x-y}{x+y} + x-y + x+y + \frac{x+y}{x-y} = a \quad \left. \vphantom{\frac{x-y}{x+y} + x-y + x+y + \frac{x+y}{x-y} = a} \right\} \text{by the question.}$$

$$\text{and } (x-y)^2 \times (x+y)^2 = b$$

From the second of these equations, by putting $c = \sqrt{b}$, and extracting the square root, there comes out

$$x-y \times x+y (=xx-yy) = c,$$

Moreover, from the first, we get

$$(x-y)^3 + 2x \times x-y \times x+y + (x+y)^3 = a \times x-y \times x+y;$$

$$\text{or, } 2cx + (x-y)^3 + (x+y)^3 = ac \text{ (by substitution)}$$

$$\text{that is, } 2cx + 2x^3 + 6xy^2 = ac;$$

$$\text{or } 2cx + 2x^3 + 6x \times xx - c = ac \text{ (because } yy = xx - c.)$$

$$\text{Hence } x^3 - \frac{cx}{2} = \frac{ac}{8} \quad \text{From which, by the theorem}$$

in page 60, the value of x may be found.

QUESTION

QUESTION CI.

The Compound Interest of a certain Sum of Money, put out for 4 Years, amounted to 344*l.* $\frac{81}{100}$; but the simple Interest thereof, for the same Time, and at the same Rate, would have been only 320*l.* What was the Sum put out? and what the Rate of Interest?

Put $a=344.81$, and $b=320$; and let x denote the interest of *1l.* for one year.

Therefore, since the simple interest of *1l.* for 4 years is $4x$, and the compound interest $(1+x)^4 - 1$, or $4x + 6x^2 + 4x^3 + x^4$,

we have, as $4x + 6x^2 + 4x^3 + x^4 : 4x :: a : b$, by the nature of the question :

and consequently $6x + 4x^2 + x^3 = \frac{4a - 4b}{b} = 0.310125$.

From the resolution of which equation, x will be found $= .05$. Therefore the rate of interest was 5 per cent. and the sum put out 1600*l.*

QUESTION CII.

The Sum (s) and the Product (p) of any two Numbers being given, to find the Sum of the Squares, Cubes, Biquadrates, &c. of those Numbers.

Let the two numbers be denoted by x and y :

then $x+y=s$
and $xy=p$ } by the question.

The former of which equations, squared, gives $x^2 + 2xy + y^2 = s^2$; from whence the double of the latter being subtracted,

we get $x^2 + y^2 = s^2 - 2p$, the sum of the squares.

Let this equation be multiplied by $x+y=s$, and there arises $x^3 + xy^2 + yx^2 + y^3 = s^3 - 2sp$,

or, $x^3 + xy \times x + y \times y^2 = s^3 - 2sp$,

that is, $x^3 + p \times s + y^3 = s^3 - 2sp$, (because $xy=p$, and $x+y=s$).

Therefore

Therefore $x^3 + y^3 = s^3 - 3sp$, the sum of the cubes.
 Again, multiply this last equation by $x + y = s$,
 and you will have $x^4 + xy \times x^3 + y^4 + y^3 = s^4 - 3s^2p$,
 or $x^4 + p \times s^3 - 2p + y^4 = s^4 - 3s^2p$ (because $x^2 + y^2 = s^2 - 2p$).
 Whence $x^4 + y^4 = s^4 - 4s^2p + 2p^2$, the sum of the fourth powers.

Multiply, again, by $x + y = s$, and you will have
 $x^5 + xy \times x^4 + y^5 + y^4 = s^5 - 4s^3p + 2sp^2$,
 or, $x^5 + p \times s^4 - 3sp + y^5 = s^5 - 4s^3p + 2sp^2$,
 and therefore $x^5 + y^5 = s^5 - 5s^3p + 5sp^2$, the sum of the fifth powers.

From whence the law of continuation is manifest; being such, that the sum of the next superior powers will, always, be obtained by multiplying the sum of the powers last found by s , and subtracting the sum of the preceding ones drawn into p , from the product. So that the sum of the n^{th} powers (expressed in a general manner)

$$\text{or } x^n + y^n, \text{ will be } = s^n - ns^{n-2}p + n \times \frac{n-3}{2} s^{n-4} p^2 - n \times \frac{n-4}{2} \times \frac{n-5}{3} s^{n-6} p^3 + n \times \frac{n-5}{2} \times \frac{n-6}{3} \times \frac{n-7}{4} s^{n-8} p^4 - \&c.$$

QUESTION CIII.

The Sum (a) and the Sum of the Squares (b) of four Numbers, in Geometrical Progression being given; to find the Numbers.

If x and y be assumed to represent the two middle numbers, then, from the nature of continued proportionals, the two extremes will be expressed by $\frac{xx}{y}$ and $\frac{yy}{x}$: and so

$$\text{we shall have } \frac{xx}{y} + x + y + \frac{yy}{x} = a,$$

$$\text{and } \frac{x^4}{y^2} + x^2 + y^2 + \frac{y^4}{x^2} = b.$$

Put

Put $x+y=u$, and $xy=z$:

then, from the first equation, $\frac{x^2}{y} + \frac{y^2}{x} = a-u$,

and therefore $x^3+y^3 = xy \times a-u$.

But, because the sum of the two means (x and y) is expressed by u , and their rectangle by z , it is evident, from question 102, that the sum of their squares (x^2+y^2) will be exhibited by u^2-2z , and the sum of their cubes (x^3+y^3) by u^3-3uz .

And, in like manner, the sum of the two extremes ($\frac{x^2}{y} + \frac{y^2}{x}$) being denoted by $a-u$, and their rectangle by z , the sum of their squares will be $=a-u^2-2z$, and the sum of their cubes $=a-u^3-3a-3u \times z$ (but this last is of no use in the present case.)

Hence, by substituting these several values in our second, and last equations,

we get $a-u^2-2z+u^2-2z=b$,

and $u^3-3uz=z \times a-u$;

that is, by reduction,

$$2u^2-2au+a^2-b=4z,$$

$$\text{and } u^3=2u+a \times z;$$

and, consequently, $2u+a \times 2u^2-2au+a^2-b=4u^3$;

whence, by reduction, $u^2+\frac{bu}{a}=\frac{aa-b}{2}$;

$$\text{and therefore } u=\sqrt{\frac{aa-b}{2}+\frac{bb}{4aa}-\frac{b}{2a}}.$$

From which the several values of z , x , y will also become known.

QUESTION CIV.

The Sum (a) and the Sum of the Cubes (c) of four Numbers in continued Geometrical Proportion being given; to determine the Numbers.

Let the notation of the preceding problem be retained; then our two equations (in this case)

K

will

will be $\frac{x^2}{y} + x + y + \frac{y^2}{x} = a$,

and $\frac{x^6}{y^3} + x^3 + y^3 + \frac{y^6}{x^3} = c$.

But, it appears, *from thence*, that the sum of the cubes $(x^3 + y^3)$ of the two means is $= u^3 - 3ux$; and that the sum of the cubes $(\frac{x^3}{y} + \frac{y^3}{x})$ of the two extremes is $=$

$\overline{a-u}^3 - 3a - 3u \times x$. Therefore our last equation, by substituting these values, becomes $u^3 + \overline{a-u}^3 - 3az = c$.

And, it appears, *by the last problem*, that the first equation (by a like substitution) is reduced to $u^3 = 2u + a \times x$.

Hence, by exterminating z out of the two equations thus derived, and putting, $\frac{a^2}{3} - \frac{c}{3a} = d$, we obtain

$$\overline{2u + a \times u^2 - au + d} = u^3;$$

$$\text{or } u^3 - au^2 - aa - 2d \times u + ad = 0.$$

From whence, the value of u being found, the rest of the quantities will be known.

In the same manner the problem may be resolved, when (instead of the sum of the cubes) the sum of the 4th, 5th, or 6th, &c. powers is given.

For the sum of the n^{th} powers of the two means (or $x^n + y^n$) being, *universally*, $= u^n - nu^{n-2}x + n \times \frac{n-3}{2} u^{n-2}x^2$ &c. (See Question 102.)

and the sum of the n^{th} powers of the two extremes

$$= \overline{a-u}^n - n \times \overline{a+u}^{n-2}x + n \times \frac{n-3}{2} \overline{a-u}^{n-2}x^2, \text{ \&c. (since the sum of the roots is here } = a-u);$$

$$\text{we therefore have } u^n + \overline{a-u}^n - nx \times u^{n-2} + \overline{a-u}^{n-2}x + \frac{n}{1} \times \frac{n-3}{2} x^2 \times u^{n-4} + \overline{a-u}^{n-4} - \text{\&c.} = c. \text{ Which}$$

equation, by writing $\frac{u^3}{2u+a}$ instead of its equal x , becomes

comes $u^n + \overline{a-u}^n - \frac{nu^{\frac{n-1}{2}}}{2u+a} \times \overline{u^{n-2} + a-u}^{n-2} + \frac{n \times n-3}{2}$
 $\times \frac{u^{\frac{n-5}{2}}}{2u+a} \times \overline{u^{n-4} + a-u}^{n-4} + \text{Ec.} = c.$ Whence u may
 be found.

QUESTION CV.

Having given the Sum (a) and the Sum of the Squares
 (b) of five Numbers in Geometrical Progression; to
 determine the Progression.

Let x , z , and y denote the three middle numbers,
 taken in order: then $\frac{xx}{z}$ will be the first number,
 and $\frac{yy}{z}$ the last; and we shall have

$$\left\{ \begin{array}{l} \frac{xx}{z} + x + z + y + \frac{yy}{z} = a, \\ \frac{x^4}{z^3} + x^2 + z^2 + y^2 + \frac{y^4}{z^3} = b, \end{array} \right\} \text{by the question.}$$

Put $u = x + y$; then, from the first equation,

$$\frac{xx}{z} + \frac{yy}{z} = a - u - z.$$

Therefore, seeing the sum of the two extremes is
 expressed by $a - u - z$, and their rectangle by z^2 (from
 the nature of proportionals) the sum of their squares
 will be $= a - u - z)^2 - 2z^2$ (by Question 102.)

Moreover, the sum $(x + y)$ of the two terms adjacent
 to the middle one being $= u$, and their rectangle $=$
 z^2 , the sum of their squares $(x^2 + y^2)$ will therefore be
 $= u^2 - 2z^2$ (by the same.)

And so, by substituting these values above,
 we get $a - u - z)^2 - 2z^2 + u^2 - 2z^2 + z^2 = b$,

and $\frac{u^3 - 2z^3}{z} = a - u - z.$

Whence $a^2 - 2au - 2az + 2u^2 + 2uz - 2z^2 = b$,
and $az - u^2 - uz + z^2 = 0$.

The double of which last equation, added to the former, gives $a^2 - 2au = b$; whence $u = \frac{a}{2} - \frac{b}{2a}$.

From which, and the equation $az - u^2 - uz + z^2 = 0$, the value of z will also become known.

QUESTION CVI.

The Sum (a) and the Sum of the Cubes (b) of five Numbers, in continued Geometrical Proportion being given; to find the Numbers.

Retaining the notation of the last problem, and proceeding in the same manner, we have

$$a - u - z = 1^3 - 3a - 3u - 3z \times z^2 + u^3 - 3uz^2 + z^3 = b;$$

and $az - u^2 - uz + z^2 = 0$ (as before.)

The first of which equations, by reduction, becomes

$$a^3 - 3a^2 \times u + z + 3a \times u^2 + 2uz - 3uz^2 + 3z^2 = b;$$

From whence the other equation, multiplied by $3z$, being subtracted, there remains

$$a^3 - 3a^2 \times u + z + 3a \times u^2 + 2uz - z^2 = b;$$

$$\text{therefore } u^2 + 2uz - z^2 - a \times u + z = \frac{b}{3a} - \frac{a^2}{3}.$$

But, by the second equation, $u^2 + 2uz - z^2 = az + uz$;

$$\text{whence, by substitution, } az + uz - a \times u + z = \frac{b}{3a} - \frac{a^2}{3};$$

$$\text{that is, } uz - az = \frac{b}{3a} - \frac{a^2}{3};$$

$$\text{or } az - uz = d \text{ (by putting } \frac{aa}{3} - \frac{b}{3a} = d).$$

From which, the second equation being subtracted, there results

$u^2 - z^2 = d$: wherein let $\left(\frac{az-d}{z}\right)$ the value of u (found from the former equation) be now substituted, and

and we shall have $\frac{ax-d}{z^2} - z^2 = d$; and consequently

$$z^4 - aa - d \times z^2 + 2adz - d^2 = 0.$$

Whence z will be found.

QUESTION CVII.

The Sum (a), the Sum of the Squares (b), and the Sum of the Cubes (c), of any three Numbers being given; to determine the Numbers.

Let them be denoted by x , y and z ; then

since $\left\{ \begin{array}{l} x + y + z = a \\ x^2 + y^2 + z^2 = b \\ x^3 + y^3 + z^3 = c \end{array} \right\}$ we shall, by transposition, have

$$x + y = a - z$$

$$x^2 + y^2 = b - z^2$$

$$x^3 + y^3 = c - z^3.$$

Now, by multiplying together the two first of these equations,

$$\text{we have } x^3 + x^2y + xy^2 + y^3 = ab - bz - az^2 + z^3.$$

And, by cubing of the first, we also have

$$x^3 + 3x^2y + 3xy^2 + y^3 = a^3 - 3a^2z + 3az^2 - z^3: \text{ which, de-}$$

ducted from the treble of the former,

$$\text{leaves } 2x^3 + 2y^3 = 3ab - a^3 + 3a^2z - 3bz - 6az^2 + 4z^3:$$

and, this being $= 2c - 2z^3$ (by the third equation) we

therefore have

$$6z^3 - 6az^2 + 3a^2 - 3b \times z = a^3 - 3ab + 2c,$$

$$\text{and consequently } z^3 - az^2 + \frac{a^2 - b}{2} \times z = \frac{a^3 - 3ab + 2c}{6}.$$

From whence (when a , b and c are expressed in numbers) three different roots, or values of z , may be found, answering all the conditions of the problem.

Thus, for example, let $a=9$, $b=29$, and $c=99$; then our equation will become $z^3 - 9z^2 + 26z - 24 = 0$.

And (by either of the two first methods explained in Sect. 12. of my *Treatise of Algebra*) the three roots, in this case, will be found to be 2, 3 and 4. which

Which numbers are, therefore, the true values of x , y and z , in the equations $x+y+z=9$, $x^2+y^2+z^2=29$, and $x^3+y^3+z^3=99$.

QUESTION CVIII.

The Sum (a), the Sum of the Squares (b), the Sum of the Cubes (c), and the Sum of the Biquadrates (d), of any four Numbers being given; to determine the Numbers.

Let the four Numbers be denoted by x , y , z and u ; and put $A=a-u$, $B=b-u^2$, $C=c-u^3$, and $D=d-u^4$. From whence, by the conditions of the problem,

$$x + y + z = A,$$

$$x^2 + y^2 + z^2 = B,$$

$$x^3 + y^3 + z^3 = C,$$

$$x^4 + y^4 + z^4 = D.$$

Now, if the Second of these equations, be subtracted from the square of the first, we shall have $2xy+2xz+2yz=A^2-B$.

And if, in like manner, the fourth equation be subtracted from the square of the second, we shall have $2x^2y^2+2x^2z^2+2y^2z^2=B^2-D$.

Moreover, if from the square of the former of these last equations, the double of the latter be deducted, there will come out

$$8x^2yz+8y^2xz+8z^2xy=A^4-2A^2B-B^2+2D;$$

$$\text{or, } 8xyz \times x+y+z = A^4-2A^2B-B^2+2D;$$

$$\text{whence } xyz = \frac{A^4-2A^2B-B^2+2D}{8A} \quad (\text{because } x+y+z=A.)$$

Again, by multiplying the first and fifth equations into each other, we get $2x^2y+2x^2z+2y^2x+2y^2z+2x^2x+2z^2y+6xyz=A^3-AB$.

And, by multiplying the first and third together, there arises $x^3+y^3+z^3+x^2y+x^2z+y^2x+y^2z+z^2x+z^2y=AB$.

The

The double of which last taken from the precedent, leaves
 $6xyz - 2x^3 - 2y^3 - 2z^3 = A^3 - 3AB$. And this, added to,
 $2x^3 + 2y^3 + 2z^3 = 2C$,
 gives $6xyz = A^3 - 3AB + 2C$.

Hence $\frac{A^3 - 3AB + 2C}{6} (=xyz) = \frac{A^4 - 2A^2B - B^2 + 2D}{8A}$

(p. above.)

and consequently, by reduction, $A^4 - 6A^2B + 8AC + 3B^2 - 6D = 0$.

In which equation let the several values of A, B, C and D, be now substituted, and then (dividing the whole

by 24) we, at length, have $u^4 - au^3 + \frac{a^2 - b}{2} u^2 - \frac{a^3 - 3ab + 2c}{6} u + \frac{a^4 - 6a^2b + 8ac + 3b^2 - 6d}{24} = 0$: whose

four roots (found by any of the known methods) answer all the conditions of the problem.

QUESTION CIX.

To find the least Whole Number, which being divided by 19, shall produce a Remainder of 7; but, being divided by 28, the Remainder shall be 13.

Let $19x + 7$, denote the number sought; where x , according to the question, must be a whole number. And, by the question, it likewise appears that $19x + 7 - 13$, or, its equal, $19x - 6$, must be divisible by 28 (without a remainder.)

But it is plain that $28x$ is divisible by 28: therefore $(9x + 6)$ the difference between $19x - 6$ and $28x$, must also be divisible by the same number 28. For it is well known that, whatever number, or quantity, measures the whole, and one part, of another (without a remainder) must do the same by the remaining part. Hence $(18x + 12)$ the double of $9x + 6$, being divisible by 28, if the same be subtracted from $19x - 6$ (in order to get x without a coefficient) the remainder, $x - 18$, will, still, be divisible by the same number; and consequently

sequently $x-18$, either, equal to nothing, or to some multiple of 28. But, as the least value of x is required, $x-18$ must be $=0$. And therefore $19x+7=349$, the number required.

QUESTION CX.

A certain Person bought as many Geese and Ducks, together, as cost him 14 Shillings; for the Geese he paid 2s. 2d. a-piece; and for the Ducks 1s. 3d. What Number had he of each?

Let x denote the number of the geese, and y that of the ducks;

so shall $26x+15y=168$, by the question.

and therefore $y = \frac{168-26x}{15} = 11-x-\frac{11x-3}{15}$.

Which being a whole number, by the nature of the problem, $11x-3$ must, therefore, be (exactly) divisible by 15.

But it is plain that $15x$ is divisible by 15; and that its excess above $11x-3$, which is $4x+3$, must, likewise, be divisible by the same number. Let the last expression $(4x+3)$ be now multiplied by 3, and the preceding one $(11x-3)$ subtracted from the product (in order to get x without a coefficient) whence you will have $x+12$; which being, still, divisible by 15, it is plain that x must either be 3, or 3 added to some multiple of 15, as 18, 33, 48, &c. But it is apparent, from the nature of the question, that all these numbers, except the first, are too large. Therefore there were 3 geese, and 6 ducks; which last number (the value of x being known) is found directly from the equation exhibited above.

QUESTION

QUESTION CXI.

One having, at Play, won a certain Number of Guineas, not exceeding 100, and being asked to tell the Number, made this Reply: "If the Number of Guineas I have won be divided by 9, there will remain 6; but, if the Number of Shillings contained in them be divided by 39, there will remain 12." The Question is, to find what Number of Guineas he was a Winner of.

Let $9x+6$ denote the number sought; where, according to the question, x must be a whole number. Then, the number of shillings being $189x+54$, it also appears that $\frac{189x+126-12}{39}$, or its equal $4x+2+\frac{11x+12}{13}$ must be a whole number; and therefore $11x+12$, divisible by 13.

Let the number 12 (for the sake of brevity) be denoted by n : then, $13x$, and $11x+n$ being, both, divisible by 13, their difference $2x-n$ must also be divisible by the same number; and so, likewise, $2x-n \times 5$, or its equal $10x-5n$. And, if this be subtracted from $11x+n$, the remainder $x+6n$ (or $x+72$) will, still, be divisible by 13. But $\frac{72}{13}$ is $=5+\frac{7}{13}$: therefore $x+7$ must be divisible by 13; and consequently the value of x , either, equal to 6, or 6 added to some multiple of 13: but, as the value of $9x+6$ is not to exceed 100 (by the question) that of x cannot be greater than 6. And therefore the number sought can be no other than 60.

QUESTION CXII.

A Person, in exchange for a Number of Pieces of Foreign-Gold, valued at 17s. 4d. each, received a certain Number of Guineas (not exceeding 50) and one Shilling over. What was the Sum exchanged?

If x be put for the number of pieces of foreign gold, and y for the number of guineas; then it is plain, from the question, that $52x = 63y + 3$; and consequently that

$$x = \frac{63y+3}{52} = y + \frac{11y+3}{52}.$$

Here $11y+n$ (supposing $n=3$) must be divisible by 52; as must, also, its quintuple $55y+5n$. And, if from this last, $52y$ be subtracted, and the remainder be multiplied by 4, we shall have $12y+20n$; which must be, still, divisible by the same number; and so likewise its excess $(y+19n)$ above $11y+n$. But $\frac{19n}{52} =$

$\frac{57}{52} = 1 + \frac{5}{52}$; therefore $y+5$ is, either, equal to 52, or to some multiple of it; and consequently y equal, either, to 47, 99, 151, &c.

But, as the value of y , by the question, is not greater than 50, all the numbers, after the first, are too large. Hence it appears that he received 47 guineas and one shilling, in exchange for 57 foreign-pieces; amounting in value to 49l. 8s. sterling.

QUESTION CXIII.

One laid out 10 Shillings in 20 Fowls, of three different Sorts, viz. Chickens, Pigeons, and Larks: the Chickens cost him 12d, the Pigeons 4d; and the Larks 1d, a-piece. How many had he of each?

Let x , y , and z denote the numbers of the three several sorts, respectively.

Then

Then will $\begin{cases} x+y+z=20 \\ 12x+4y+z=120 \end{cases}$ by the question.

And, by subtracting the former of these equations from the latter, we have $11x+3y=100$; and therefore $y=$

$$\frac{100-11x}{3} = 33 - 3x - \frac{2x-1}{3}.$$

Now, $2x-1$ being divisible by 3, it is evident that $(x+1)$ its difference from $3x$, must likewise be divisible by 3; and, consequently, that x must either be 2, or 2 increased by some multiple of 3; that is, equal to some one of the numbers 2, 5, 8, 11, 14, &c. But, as neither y nor z can be greater than 18 (by the question) so all the foregoing numbers, below and above 8, give the value of y either too great, or too small.

But, when x is taken 8, y will come out $=4$, and $z=8$; which are the three numbers required.

QUESTION CXIV.

To determine all the several Ways whereby it is possible to pay 60l. in Guineas and Moidores, only.

Let x denote the number of guineas, and y the number of moidores.

Then will $21x+27y=1200$; or $7x+9y=400$, by the problem; and therefore $x = \frac{400-9y}{7} = 57 - y -$

$\frac{2y-1}{7}$. From whence, as $2y-1$ is divisible by 7, it will appear, by reasoning as in the preceding examples, that $y+3$ must be divisible by the same number; and consequently that the least value of y is $=4$.

and the corresponding value of $x (=57 - y - \frac{2y-1}{7}) = 52$.

Now, having found the least value of y , and the greatest of x , the rest of the answers will be obtained, by adding 7 (the coefficient of x in the above equation) to the last value of y , continually, and subtracting 9 (the coefficient of y) from the last value of x . By

means of which we get the 6 following solutions, being all the question admits of.

$$\text{viz. } \begin{cases} x=52, 43, 34, 25, 16, \text{ or, } 7, \\ y=4, 11, 18, 25, 32, \text{ or, } 39. \end{cases}$$

QUESTION CXV.

To find how many Ways it is possible to pay 20l. in Half-Guineas, and Half-Crowns, without any other Sort of Coin.

If x be the number of half-guineas, and y the number of half-crowns; we shall have $21x+5y=800$; and therefore $y=\frac{800-21x}{5}=160-4x-\frac{x}{5}$. Whence it appears, at one view, that x is a multiple of 5: and therefore, the several, required, values of x being expressed by 5, 10, 15, 20, 25, 30 and 35, those of y , answering thereto, must be 139, 118, 97, 76, 55, 34, and 13, respectively. So that there are 7 answers in this case,

QUESTION CXVI.

A Reckoning of 20 Shillings was spent by a Company of twenty Persons, consisting of Officers, Sailors, and Marines: each Officer paid 2s. 6d. each Sailor 12d, and each Marine 8d. How many Persons were there of each Denomination?

Let the three required numbers be denoted by x , y , and z , respectively;

so shall $\begin{cases} x+y+z=20 \\ 30x+12y+8z=240 \end{cases}$ by the question.

And, by subtracting 8 times the former of these equations from the latter, we shall have $22x+4y=80$;

and therefore $y=20-\frac{11x}{2}$.

But, y being a whole number, it is plain that x must be an even number, and, also, less than 4; and therefore

fore can be no other than 2. From whence y is given $=9$, and $z=9$, likewise.

QUESTION CXVII.

To find a Number, which, being divided by 28, shall produce a Remainder of 19; but, being divided by 19, the Remainder shall be 15; and, being divided by 15, the Remainder shall be 11.

Let $28x+19$ denote the number sought; where x , according to the first condition of the problem, must be a whole number. And, by the second condition, it appears that $28x+19-15$ must be divisible by 19. Whence (following the method observed in the preceding examples) the least value of x is found $=8$: and so $19z+8$ (where z denotes any whole number) is a general value of x , answering the two first conditions.

Let this be, therefore, substituted instead of x ; and our assumed expression will become $532z+243$. From whence, as $532z+243$ is divisible by 15, the least value of z will be found $=14$. And $15n+14$, will be a general value of z : which, substituted in $532z+243$, gives $7980n+7691$ for a general answer to the problem; where n may be, either, equal to nothing, or any whole number.

QUESTION CXVIII.

To find three Numbers, in the Proportion of 5, 7, and 9; which being, severally, divided by 11, 13, and 15, there shall remain 1, 2, and 3, respectively.

Let $5x$, $7x$, and $9x$ denote the three required numbers: then, by the question, $5x-1$, $7x-2$, and $9x-3$, must be, respectively, divisible by 11, 13, and 15 (without leaving any remainder.)

But it will be found (by proceeding as in the foregoing problems) that the least value of x to answer the first of these conditions, will be $=9$: therefore $9+11z$
(where

(where z denotes any whole number) will be a general value of x , answering the same condition.

Let this value be, therefore, substituted in the second and third expressions; which, by that means, will become $77z+61$, and $99z+78$. And then, as the former of these is divisible by 13, the least value of z , to fulfil this condition, will (also) be found $=9$.

Let, therefore, $9+13u$ (which is a general value for z) be substituted in the last of the three expressions, and it will become $13u+9 \times 99+78$. Which being divisible by 15, the $\frac{1}{3}$ part thereof, or $13u+9 \times 33+26$ ($=429u+323$) must, consequently, be divisible by 5. Whence u is found $=3$: therefore z ($=13u+9$) $=48$, and x ($=11z+9$) $=537$. So that the three least numbers, answering the conditions of the problem, are 2685, 3759, and 4833.

QUESTION CXIX.

Supposing $6x+7y+8z=100$; it is required to find all the possible values of x , y , and z , in whole Numbers.

In questions of this kind, where you have three, or more, indeterminate quantities, and but one equation, it will be proper, first of all, to find the limits of those quantities. Thus, because $x = \frac{100-7y-8z}{6} = 16-y-z -$

$\frac{y+2z-4}{6}$, it appears that x cannot be greater than 14.

And, in the same manner, it will appear that y cannot be greater than 12; nor z greater than 10.

Now, as x is a whole number, by the question, $y+2z-4$ must therefore be divisible by 6: and, as $2z$ and 4 are even numbers, it is plain that y must also be an even number (since an odd one cannot be divided by an even one, without a remainder). Let y be, therefore, first expounded by the least even number (2), so will $y+2z-4$ become $=2z-2$: which, being divisible by 6, it is plain that $z-1$ (the half thereof) is divisible

divisible by 3; and consequently that the several values of z (when $y=2$) are 1, 4, 7, and 10.

Whence the corresponding values of x , by substituting above, will appear to be 13, 9, 5, and 1.

Let y be now taken $=4$; then $y+2z-4$ will be $=2z$; and so, z being divisible by 3, the several values of z , in this case, will be 3, 6, 9. But the two first of these, only, are for our purpose, the last giving $x=0$.

By taking $y=6$, and proceeding in the same manner, we shall get two other answers; wherein z will be 2, and 5; and x , 7 and 3. And, by taking $y=8$, two more answers will be found (making 10 in the whole) which are all the question admits of; and which, being placed in order, will stand as below.

y	z	x
2	1. 4. 7. 10	13. 9. 5. 1
4	3. 6.	8. 4.
6	2. 5.	7. 3.
8	1. 4.	6. 2.

QUESTION CXX.

If $17x+19y+21z=400$; it is proposed to find all the possible Values of x , y and z , in whole positive Numbers.

When the coefficients of the indeterminate quantities x , y and z are nearly equal, as in this example, it will be convenient to substitute for the sum of those quantities: thus, let $x+y+z=m$; then, by subtracting 17 times this last equation from the preceding one, we shall have $2y+4z=400-17m$; and by subtracting the given equation from 21 times the assumed one $x+y+z=m$, there will remain $4x+2y=21m-400$. Therefore, since y and z can have no values less than unity, it is plain, from the first of these two equations, that $400-17m$ cannot be less than 6, and therefore m

not

not greater than $\frac{400-6}{17}$ or 23: also, because by the second of the two last equations, $21m-400$ cannot be less than 6, it is obvious that m cannot be less than $\frac{400+6}{21}$ or 19: therefore 19 and 23 are the limits of m in this case. These being determined, let $4x$ be transposed in the last equation, and the whole divided by 2, and we shall have $y = 10m - 200 - 2x + \frac{m}{2}$; which being a whole number, by the question, it is evident that $\frac{m}{2}$ must likewise be a whole number, and consequently m equal to an even number; which, as the limits of m are 19 and 23, can only be 20, or 22: Let, therefore, m be first taken = 20, then y will become $= 10 - 2x$ and $z(m-x-y) = 10 + x$; wherein x being taken equal to 1, 2, 3 and 4 successively, we shall have y equal to 8, 6, 4, 2, and z equal to 11, 12, 13, 14, respectively; which are four of the answers required. Again, let m be taken = 22, then will $y = 31 - 2x$ and $z = x - 9$, in which let x be taken equal to 10, 11, 12, 13, 14 and 15 successively, and y will come out = 11, 9, 7, 5, 3 and 1, and $z = 1, 2, 3, 4, 5$ and 6, respectively. Therefore we have the ten following answers in whole numbers; which are all the question admits of:

$$\begin{array}{l} x = 1 \mid 2 \mid 3 \mid 4 \mid 10 \mid 11 \mid 12 \mid 13 \mid 14 \mid 15 \\ y = 8 \mid 6 \mid 4 \mid 2 \mid 11 \mid 9 \mid 7 \mid 5 \mid 3 \mid 1 \\ z = 11 \mid 12 \mid 13 \mid 14 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6. \end{array}$$

QUESTION

QUESTION CXXI.

To find two Whole Numbers, whereof the Difference of the Squares shall be 77.

Let the leffer number be x , and the greater $x+m$; and suppose the number given to be represented by a : so shall $(x+m)^2 - x^2 = a$, that is, $2mx + mm = a$:

and consequently $x = \frac{a - mm}{2m} = \frac{a}{2m} - \frac{m}{2}$.

Whence we also have $x+m = \frac{a}{2m} + \frac{m}{2} \left(= \frac{a+mm}{2m} \right)$.

But, in the case proposed, a being = 77.

x becomes = $\frac{77}{2m} - \frac{m}{2}$, and $x+m = \frac{77}{2m} + \frac{m}{2}$:

which being both required in whole numbers, it is evident, in the first place, that m must be some divisor of 77; and, secondly, that $\frac{77}{m}$ must be greater than m ; and consequently m less than 9.

But the divisors of 77, below 9, are 1 and 7: which numbers being wrote successively, in the room of m , the corresponding values of x will come out 33, and 2; and those of $x+m$, 34 and 9, respectively. So that the question, in the case proposed, admits of two answers, and no more.

QUESTION CXXII.

To find a Whole Number, to which 12 and 25 being, successively, added, both the Sums shall be square Numbers.

Let z be the number sought; and assume x and $x+m$ for the roots of the two squares:

then will $\left\{ \begin{array}{l} z+12 = x^2 \\ z+25 = (x+m)^2 \end{array} \right\}$ by the question.

M

Hence,

Hence, by subtracting the former equation from the latter, we get $13 = \overline{x+m}^2 - x^2 = 2mx + mm$;

and therefore $x = \frac{13}{2m} - \frac{m}{2}$. Which being a whole number, m must be $= 1$: whence $x = 6$; and $z (=x^2 - 12) = 24$.

QUESTION CXXIII.

To find three Whole Numbers, so that the Sum of the Squares of the two least of them shall be equal to the Square of the Greatest.

It appears from the problem preceding the last, that the difference of the squares of $\frac{a+mm}{2m}$, and $\frac{a-mm}{2m}$ is, universally, equal to a ,

or $\frac{\overline{a+mm}^2}{4mm} - \frac{\overline{a-mm}^2}{4mm} = a$; let a and m be what they will.

Whence it is also plain that $\overline{a+mm}^2 = \overline{a-mm}^2 + 4mma$.

But, since it is required to have $4mma$, a square number (as well as the other two) a must therefore be a square number; let it be n^2 , and then our equation will become $\overline{nn+mm}^2 = \overline{nn-mm}^2 + 2mn^2$: where m and n may be expounded by any whole numbers, at pleasure.

Thus, for example, suppose $m=1$, and $n=2$; then there will come out $5^2 = 3^2 + 4^2$. Again, let $m=2$, and $n=3$, and there arises $13^2 = 5^2 + 12^2$. Lastly, let $m=2$ and $n=5$, and you will get $29^2 = 21^2 + 20^2$.

QUESTION

QUESTION CXXIV.

To find three Whole Numbers, whose Squares are in Arithmetical Progression.

Let x , $x+m$, and $x+n$ express three such numbers. So shall $x+m)^2 - x^2 = x+n)^2 - x+m)^2$, by the nature of the problem.

Whence x is found $= \frac{nn-2mm}{4m-2n}$.

Put $4m-2n=a$, and $n^2-2m^2=b$; then $x = \frac{b}{a}$, $x+m = \frac{b+am}{a}$, and $x+n = \frac{b+an}{a}$. Now, as the squares

of $\frac{b}{a}$, $\frac{b+am}{a}$, and $\frac{b+an}{a}$ are in arithmetical progression, it is plain that the squares of their *equimultiples*, b , $b+am$, and $b+an$, must be in arithmetical progression likewise. From whence, by expounding m and n by different whole numbers, successively, as many particular answers as you please, may be exhibited.

Thus, if $m=2$ and $n=3$; then, a being $=2$, and $b=1$, there will come out 1, 5, and 7. But, if $m=3$ and $n=5$, we shall get 7, 13, and 17, for another answer.

QUESTION CXXV.

Supposing $x^2 = z^2 + az + b$ (where a and b denote given Numbers); it is required to find the Values of x and z (if possible) in Whole Numbers.

Put $x=z+m$; then, by substitution.

$$z^2 + 2mz + m^2 = z^2 + az + b;$$

and consequently $z = \frac{m^2 - b}{a - 2m}$. Which value, by putting

$$a - 2m = n \text{ (or } m = \frac{a - n}{2} \text{) becomes } = \frac{a^2 - 2an + n^2 - 4b}{4n} = \frac{1}{4} \text{ of}$$

$\frac{1}{4}$ of $\frac{aa-4b}{n} - 2a+n$. From which it is evident, that, to have the value of z a whole number, n must be some divisor of the given quantity $aa-4b$, and therefore even or odd, according as a is even or odd.

Example. Suppose the given equation to become $x^2 = z^2 + 20z$; in which case, a being $=20$, and $b=0$, we have $z = \frac{1}{4} \times \frac{400}{n} - 40 + n$; where the, even, divisors of 400 are 2, 4, 8, 10, &c. whereof the second will be found to answer; the values of z and x coming out 16 and 24, respectively.

Again, suppose the given equation to become $x^2 = z^2 + 100z + 1000$: here we have $z = \frac{1}{4} \times \frac{6000}{n} - 200 + n$: And the, even, divisors of 6000, are 2, 4, 6, 8, 10, 12, 16, 20, &c. Whereof 4, 12, and 20 succeed. By the last of these (which determines the least values) z comes out $=30$, and $x=70$.

QUESTION CXXVI.

Having given $x^2 = a^2 + bz + cz^2$, wherein a , b and c denote given Whole Numbers; it is required to find the Values of x and z (if possible) in Whole Numbers.

Put $x = a + mz$; then will $(a + mz)^2 = a^2 + bz + cz^2$; that is, $a^2 + 2amz + m^2z^2 = a^2 + bz + cz^2$.

Whence z comes out $= \frac{b - 2am}{mm - c}$:

where it is evident, that, in order to have a positive value, m must be taken equal to some number between $\sqrt{c}$, and $\frac{b}{2a}$.

Thus, supposing the given equation to become $x^2 = 64 - 12z + 5z^2$, the value of m , in this case, must be less

less than $\sqrt{5}$, and greater than $-\frac{12}{16}$. Let it therefore be expounded by 2 and 1, successively; whence $\frac{-12-16m}{mm-5}$, or its equal $\frac{12+16m}{5-mm}$ (which is, here, the value of z) will come out 44 and 7 respectively; and the corresponding values of x ($a+mx$) are found to be 96 and 15: both which answer the conditions of the problem.

PART

PART II.

CONTAINING

*A Variety of GEOMETRICAL PROBLEMS, with
their SOLUTIONS: both by ALGEBRA, and
independent of it, from Principles purely GEOME-
TRICAL. NB. if Author quotes his own Elements
of Geometry 4th Edit. sup. 89. p. m. n.*

ALTHOUGH the chief design of this work consists not in exhibiting a number of rules and precepts, but in illustrating those already known, on the subject, by a set of proper and useful examples; yet, as we are in this part to treat of the investigation of Geometrical Problems, without calling in the assistance of Algebra (a thing hitherto very little considered by authors, though in itself very interesting and useful) it may not be amiss, before we proceed to particular cases, to premise a few general observations on this head.

1. In the first place, then, it is necessary, in order to the construction of Geometrical Problems, that something of the *Geometric Loci* should be understood.— Thus it will be of use to know that the place of the vertex of a triangle, whereof the base and opposite angle are supposed to remain constant, while the other sides and angles vary, will *always* fall in the circumference of a circle passing through the extremities of the base. This appears from *Euclid, B. 3. Prop. 21*; and is also demonstrated in my *Elements of Plane Geometry, B. 3, Prop. 9, 1st edit. or B. 3, Theor. 11, 4th edit.*

It ought moreover to be known, that the locus of the vertex, when the ratio of the two sides and the base

base of the triangle are given, or continue invariable, will, *also*, be the circumference of a circle, dividing the base in the given ratio. (*For the demonstration of which, see Elem. Plane Geometry, B. 4, Prop. 15, 1st edit.*)

If the sum, or the difference, of the squares of the two sides, together with the base, be supposed given, the locus of the vertex will, *still*, be the circumference of a circle, in the former case; and a right-line, in the latter (*vid. B. 2, Prop. 12, 1st edit. or B. 2, Theor. 11, 4th edit.*)

But, if the sum, or the difference, of the sides, themselves, is given; then the locus will, either, be the arch of an ellipsis or an hyperbola (as is well known to those who have touched upon conic-sections.) But these two last are not admitted in the construction of linear and plane problems; of which, only, we purpose to treat.—The problems, of the following collection, wherein the use of the *Geometric-Loci*, above specified, is particularly exemplified, are the 11, 12, 19, 34, 35, 48, and 52^d.

2^o When, in the figure to be constructed, the sum, or the difference, of two adjacent sides happens to be given; it will be proper, first, to form a triangle, so that the said sum, or difference, may be one of its sides; and, then, to consider, what other sides, or angles, will be given, or become known, in consequence thereof. This rule is illustrated in the 1, 2, 18, 29, 30th, and some other of the succeeding problems.

3^o It often happens that the ratio of two, or more, lines is given, from the nature of the figure, or by hypothesis, though the lines themselves are absolutely unknown: in all such cases we must endeavour, by drawing parallels (or some other way) to obtain other lines in the same given ratio: so that, one of them being given, or known from the nature of the figure or problem, the other may also become known.—The use of this rule, which is very extensive, will particularly

larly appear in the solutions to the 3, 13, 16, 22, 26, 33, 37, and 56th problems.

4^o But, if the lines, whereof the ratio is given, should happen to lye remote of each other; then, by the help of those, we must endeavour to determine the ratio of others, lying nearer together; and so on, till we obtain (if possible) the ratio of two lines, that are both sides of the same triangle; wherein one angle and the remaining side (or some other, two, parts) are given.—For the better understanding this rule, consult the solutions to the 49th and 57th problems, in particular.

5^o Lastly, when the rectangle under two unknown lines is given, either, a mean proportional must be found, or else, two other lines must be assigned, by forming similar triangles (or some other way) so as to comprehend an equal rectangle; and so that, one of them being given by the nature of the figure, the other may also become known. This rule is exemplified in the 4, 5, 6, 9, 10, 21, 38, 40, 41, 44, and 53^d problems.

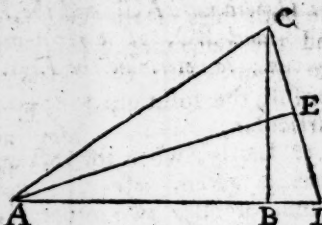
Besides the above, other observations might be here laid down; but those already delivered being the most general that have occurred to me, I shall now proceed on in the resolution of problems; the proper business of this work.

PROBLEM

PROBLEM I.

One Leg BC, and the Difference between the other Leg AB and the Hypotenuse AC, of a right-angled Triangle ABC, being given; to find both AB and AC.

Put $BC=a$, $AB=x$, and $AC=x+b$; (b being the given difference) then AC^2 being $=AB^2 + BC^2$ (Elem. 8. 2.*) we have $xx + 2bx + bb = xx + aa$: whence $2bx = aa - bb$; and consequently $x = \frac{aa - bb}{2b} = \frac{aa}{2b} - \frac{b}{2}$



$-\frac{b}{2}$. From which $AC (x+b)$ is given $= \frac{aa}{2b} + \frac{b}{2}$.

Geometrically.

If, in AB produced, there be taken BD equal to the given difference of AC and AB, and DC be drawn (according to the second general observation) it is evident that AD will be equal to AC; and the angle ACD, also, equal to the angle D.

Therefore, having taken BD as above specified, and made BC perpendicular thereto, and of the given length, and joined D, C; let CA be so drawn as to make the angle DCA=D; or, instead thereof, let a perpendicular EA be erected on the middle of CD; then the intersection of either of these lines with DB, produced, determines the triangle.

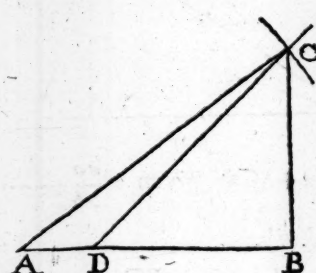
From this construction we have the very same theorem, for the numerical solution, as is derived above, from the algebraical process: for the triangles ADE and CDB, having each a right-angle, and D common,

* Note. The quotations, in this, and the succeeding problems, refer to the author's Elements of Geometry, 4th edit. printed for F. WINGRAVE.

common, are similar. Therefore $DB : DC :: DE$
 $(\frac{1}{2}DC) : AD (AC) = \frac{DC^2}{2DB} = \frac{BC^2 + BD^2}{2BD} = \frac{BC^2}{2BD} + \frac{BD}{2}$;
 as before.

PROBLEM II.

The Hypotenuse AC, and the Difference of the two Legs AB and BC, of a right-angled triangle ABC, being given; to determine the Legs.



Put $AC = a$, $AB = x$,
 and $BC = x - b$:
 so shall $xx + x - b^2 = aa$
 (Elem. 8. 2.)
 that is, $2xx - 2bx + bb = aa$.
 Whence $xx - bx = \frac{aa}{2} - \frac{bb}{2}$:
 and consequently $x =$
 $\sqrt{\frac{1}{2}aa - \frac{1}{2}bb + \frac{1}{2}b}$.

Geometrically.

If, in AB there be taken AD equal to the given difference, and CD be drawn; then, DB being = BC, the angle BDC will also be = BCD = half a right-angle.

Therefore, having laid down AD, and drawn an indefinite line DCE to make the angle BDE = $\frac{1}{2}$ right-angle; upon the center A, with the given interval AC, let an arch be described, intersecting DE in C; from which point, upon AD produced, let fall the perpendicular CB; so shall ABC be the triangle required.

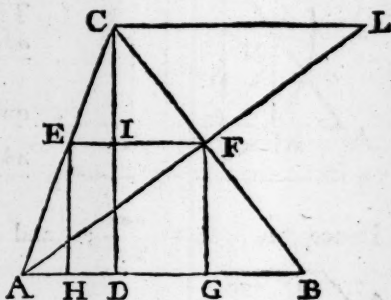
The numerical solution, according to this construction, is very easy, by the help of Trigonometry: for, two sides and one angle of the triangle ADC being given, the other angles may from thence be found; and then, all the angles and one side (AC) of the proposed triangle being known, the other sides AB and BC may also be determined.

PROBLEM

PROBLEM III.

The Base AB and the Perpendicular CD of any Triangle ABC being given; to find the Side EF, or EH, of the inscribed Square EFGH.

Put $CD=a$, $AB=b$, and $DI(=EF)=x$: then will $CI=a-x$; and, by reason of the similar triangles ABC and ECF, we shall have $a:b::a-x:x(=EF)$. Therefore $ax=ab-bx$; and consequently $x=\frac{ab}{a+b}$.



Geometrically.

The ratio of EH to EA being given, as CD to CA, EF must therefore be to EA in the same given ratio: and, if CL be drawn parallel to EF, meeting AF produced in L (*agreeable to the 3^d general observation*) it is evident, because of the similar triangles, that the line CL, so drawn, will be to CA, *still*, in the same given ratio; that is, $CL:CA::CD:CA$; and consequently $CL=CD$. Whence the method of construction is manifest.

PROBLEM IV.

To determine the Sides of a Rectangle. EFGH, inscribed in a given Triangle ABC, whose Area shall be to that of the Triangle in a given Ratio.

Put the perpendicular $CD=a$, the base $AB=b$, and the altitude EH of the rectangle $=x$; and let the given ratio of ABC to EFGH be that of m to n .

N 2

Because

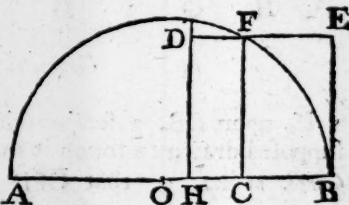
This problem, it is observable, becomes impossible when MN passes intirely without the lesser semi-circle, that is, when the given rectangle is supposed greater than half the triangle. The same thing appears also from the algebraic solution, in which case the quantity

$\left(\frac{aa}{4} - \frac{naa}{2m}\right)$ under the radical-sign, becomes negative.

PROBLEM V.

To divide a given Right-line AB into two such Parts AC and BC, that the Rectangle contained under them may be of a given Magnitude.

Put $AB=a$, and $AC=x$, and let the given magnitude, or content, of the proposed rectangle be represented by the square BD, whose side BE, or ED, let be denoted by b : then will $x \times a - x = bb$; or $xx - ax = -bb$.



Whence $x = \pm \sqrt{\frac{1}{4}aa - bb + \frac{a}{2}}$.

Geometrically.

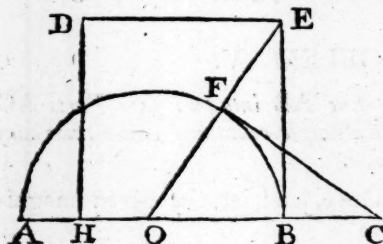
If upon AB, as a diameter, a semi-circle AFB be described, it is evident (by *Elem.* 19. 4.) that a perpendicular FC, drawn from the point wherein the circumference intersects DE, will cut AB in the point required.—It is plain, from both these solutions, that the given rectangle must not be greater than the square of half the given line, to be divided.

PROBLEM

PROBLEM VI.

To a given Line AB, it is required to add another Line BC, so that the Rectangle under the whole, compounded, Line AC, and the Part added, may be of a given Magnitude.

Let $AB=a$, $BC=x$, and the side of the given



square BEDH, expressing the magnitude of the proposed rectangle, $=b$.

Then we shall have

$a + x \times x = bb$; and consequently $x =$

$$\sqrt{bb + \frac{1}{4}aa} - \frac{1}{2}a.$$

Geometrically.

If, upon AB, a semi-circle be described, and CF be supposed drawn to touch it in D, it is plain, *from Elem. Corol. to 22. 3*, that CF^2 is $= AC \times BC = BE^2$ (*by hypothesis*); and consequently $CF = BE$: therefore, OF being $= OB$, it follows that OE and OC are likewise equal. Hence, if to the middle of AB, we draw EO, and take $OC = EO$, the thing is done.

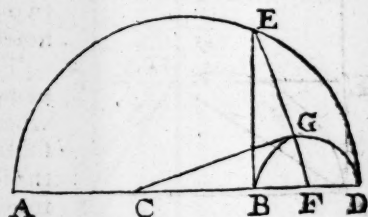
PROBLEM

PROBLEM VII.

To divide a given Right-line AB into two such Parts, that, the Rectangle under one of them AC and another, given, Line BD, may be equal to the Square of the remaining Part BC.

Put $AB=a$, $BD=b$,
and $BC=x$; then will
 $AC=a-x$,

and therefore $xx=a-x$
 $\times b$, by the question.
Hence $xx+bx=ab$;
and, consequently $x=$
 $\sqrt{ab+\frac{1}{4}bb}-\frac{1}{2}b$.



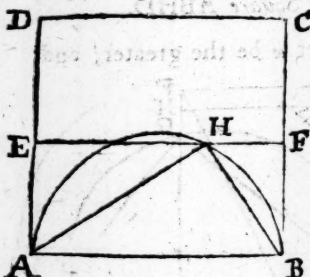
Geometrically.

Since $\overline{BC}^2=AC \times BD$ (by hypothesis) it follows, by adding $BC \times BD$ to each, that $\overline{BC}^2+BC \times BD=AC \times BD+BC \times BD$; or that, $BC \times CD=BD \times AB=\overline{BE}^2$, taking BE a mean proportional between BD and AB (by Elem. 14. 5.) But the rectangle $BC \times CD$, if a semi-circle be described upon the diameter BD, is known to be equal to the square of the tangent CG (Elem. Cor. to 22. 3). Hence $\overline{CG}^2=\overline{BE}^2$; and consequently $CG=BE$: therefore, FG being also =FB, it follows that FC is equal to FE; whence the method of construction is manifest.

PROBLEM

PROBLEM VIII.

To determine two Lines, whereof the Rectangle shall be equal to a given Rectangle ABFE, and the Sum of their Squares equal to a given Square ABCD.



Put $AB (=BC) = a$, $BF (=AE) = b$, and let the two required lines be denoted by x and y .

Then will $xy = ab$, and $xx + yy = aa$, by the question: whence by adding, and subtracting, the double of the former of these equations from the latter,

$$\text{we have } \begin{cases} xx + 2xy + yy = (x+y)^2 = aa + 2ab, \\ xx - 2xy + yy = (x-y)^2 = aa - 2ab. \end{cases}$$

$$\text{Therefore } \begin{cases} x+y = \sqrt{aa + 2ab} \\ x-y = \sqrt{aa - 2ab} \end{cases}$$

$$\text{From which } \begin{cases} 2x = \sqrt{aa + 2ab} + \sqrt{aa - 2ab} \\ 2y = \sqrt{aa + 2ab} - \sqrt{aa - 2ab}. \end{cases}$$

Geometrically.

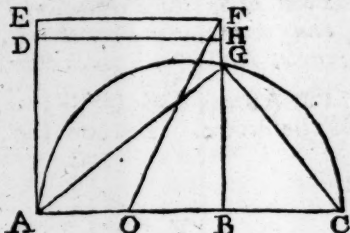
If upon AB a semi-circle be described, intersecting EF in H , then the lines joining A , H , and B , H will answer the conditions of the problem. For, the angle AHB being a right-one (*Elem. 13. 3.*) thence is $AH^2 + BH^2 = AB^2 = ABCD$ (*Elem. 8. 2.*); and $AH \times BH$ (= twice the triangle ABH) = $ABFE$ (*Elem. Corol. to 2. 2.*)

PROBLEM

PROBLEM IX.

To determine two Lines, whereof the Rectangle shall be equal to a given Rectangle ABFE, and the Difference of their Squares equal to a given Square ABHD.

Put $AB=a$, $BF=b$; and let x be the greater, and y the lesser of the two lines required: so shall $xy=ab$; and $xx-yy=aa$, by the question. From the former of which equations we have $y=\frac{ab}{x}$; which



value, substituted in the latter, gives $xx-\frac{aabb}{xx}=aa$.

Hence $x^4-a^2x^2=a^2b^2$;

and consequently $x=\sqrt{\frac{1}{2}aa+a\sqrt{bb+\frac{1}{2}aa}}$: whence y will also be known.

Geometrically.

It is evident, in the first place, that the two lines to be determined will be the hypotenuse and one leg of a right-angled triangle (ABG) whose remaining leg is the given line AB. And, since the rectangle under these lines is supposed given, another triangle ACG, similar to ABG, must therefore be assumed; so that CG, in the former, and BG, in the latter, may be homologous sides (according to the 5th general observation; vid. p. 88.)

Hence we have $CG \times AB = BG \times AG$ (Elem. 24. 3.) $= AB \times BF$ (by hyp.) and consequently $CG=BF$.

And so, $AC \times BC (=CG^2)$, Elem. Cor. 19. 4.) being given $=BF^2$, the case under consideration is now reduced to our 6th problem: whence we have the following construction.

O

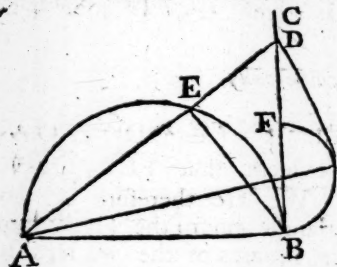
To

To the middle of AB, let FO be drawn; and, having taken OC=OF, let a semi-circle be described upon AC, cutting BF in G; so shall BG be the lesser, and AG the greater of the two lines required.

PROBLEM X.

The Diameter AB of a Semi-circle being given, to find a Point D in the Perpendicular BC, from whence DA being drawn, the Part thereof, ED, without the Semi-circle, shall be of a given length (BF.)

Put $AB=b$, $DE (=BF) = a$, and $AD=x$; also let BE be drawn. Because the angle AEB is a right-



one, the triangles ADB and ABE are similar. And therefore $AD (x) : AB (b) :: AB (b) : AE (x-a)$.

Whence $xx - ax = bb$; and consequently $x = \sqrt{bb + \frac{1}{4}aa + \frac{1}{4}a}$.

Geometrically.

Since $DA \times EA = AB^2$ (Elem. 24. 3.) where the part DE of DA is given (=BF) the case is therefore, reduced to our 6th problem.

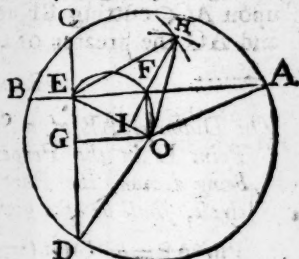
From whence it will appear, that, if upon BF a semi-circle be described, and through the center thereof, AH be drawn, meeting the periphery in H; an arch described from the center A, with the radius AH, will cut BC in the point required.

PROBLEM XI.

Having the Length of two Chords AB and CD, cutting each other at Right Angles, together with the Distance OE of the Point of their Intersection from the Center; to determine the Diameter of the Circle.

Upon the given chords, from the center O, let fall the

the perpendiculars OF and OG; and draw the radii OA and OD: also put $AF (= \frac{1}{2}AB) = a$, $DG (= \frac{1}{2}CD) = b$, $OE = c$, and $AO (= DO) = x$. Then will $OF^2 = x^2 - a^2$, and $OG^2 = x^2 - b^2$; but $OF^2 + OG^2 (= OF^2 + FE^2)$ is $= OE^2$; that is, $2x^2 - a^2 - b^2 = c^2$; and consequently $x = \sqrt{\frac{aa + bb + cc}{2}}$.



Whence the diameter is given $= \sqrt{2a^2 + 2b^2 + 2c^2}$.

Geometrically.

Since $FE^2 + DG^2$ is $(= OD^2 = OA^2) = OF^2 + AF^2$, it is evident that $FE^2 - OF^2$ is given $= AF^2 - DG^2$. We are therefore to construct a right-angled triangle upon the given hypotenuse OE, whereof the squares of the two legs shall have the same difference as the two given squares AF^2 and DG^2 .

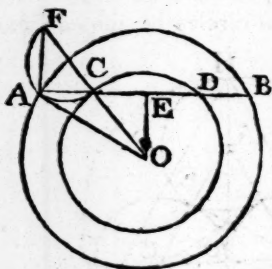
In order to which, upon OE, let a semi-circle be described; also, from the centers O and E, with radii equal to DG and AF, respectively, let two arcs be described, intersecting each other in H; from which point, upon OE, let fall the perpendicular HI; which will intersect the semi-circle in (F) the vertex of the required triangle: since it is evident that $FE^2 - OF^2 = EI^2 - OI^2 = EH^2 - OH^2 = AF^2 - DG^2$ (by construction).

Therefore if in EF produced, FA be taken of the given length, OA (when drawn) will be the radius of the required circle.

PROBLEM XII.

To draw a Right-line to cut two given concentric Circles, OAB, OCD, so that the Chords, or Parts of the said Line intercepted by those Circles, may obtain a given Ratio.

Put the radius OA of the greater circle $=a$, and the radius OC of the lesser $=b$; and let the given



ratio of AB to CD be that of m to n : then, denoting OE, the distance of AB from the center by x , we have $AE^2 = aa - xx$, and $CE^2 = bb - xx$. Therefore, AE being $=\frac{1}{2}AB$, and $CE = \frac{1}{2}CD$, it follows, that, $aa - xx : bb - xx :: m^2 : n^2$; and consequently $n^2a^2 - n^2x^2 = m^2b^2 -$

m^2x^2 : whence $x = \sqrt{\frac{mmbb - nnaa}{mm - nn}}$. By means of which AB may be drawn.

Geometrically.

Since the ratio of AE to CE is given, let OC be produced to F, so that OF may be to OC in the same, given ratio, (*agreeable to the 3^d general observation*) then, A, F being joined, the triangle CAF will be similar to the triangle CEO; and consequently the angle CAF a right one. Hence the following construction.

Having drawn the radius OC, and in it, produced, taken OF in proportion thereto, as AB is to CD (*as above specified*) let a semi-circle, upon CF be described, intersecting the greater of the two given circles in A; from which point, through C, draw AB, and the thing is done.

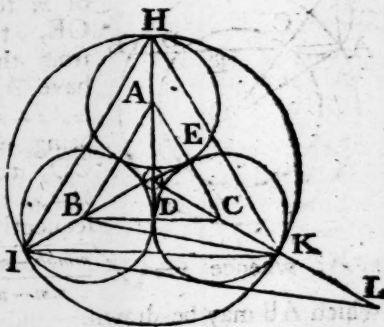
It is manifest, both from this, and the algebraical solution, that the ratio of m to n (or of AB to CD) cannot

cannot be given less than that of OA to OC, without rendering the problem impossible.

PROBLEM XIII.

To determine the Radii of three equal Circles, A, B, C, described in a given Circle HIK to touch each other and likewise the Circumference of the given Circle.

Let the centers of the several circles be joined; and let AO and BO be produced to bisect BC and AC in D and E: also let the radius (OI) of the given circle be denoted by a , and that of each of the required ones by x . Now the triangles BCE and BOD being similar, and $CE = \frac{1}{2}BC$,



it appears that OD is also $= \frac{1}{2}OB$. But $\overline{OB}^2 - \overline{OD}^2$ is $= \overline{BD}^2$; that is, in species, $\overline{a-x}^2 - \frac{\overline{a-x}^2}{4} = xx$. Which, solved, gives $x = \sqrt{12aa - 3a} = a \times 2\sqrt{3-3}$.

Geometrically.

It is evident that the right-line IK, joining the points of contact I and K, is the side of an equilateral triangle inscribed in the given circle: and, that, if in OK produced there be taken $KL = \frac{1}{2}IK$, a line drawn from I to L, will be parallel to another line drawn from B to K; because the triangles IKL and BCK (having $\angle IKL = \angle BCK$, and $IK : KL :: 2 : 1 :: BC : CK$) are equiangular.

Therefore, in order to the geometrical construction, having first drawn the radii OH, OI, and OK, to divide

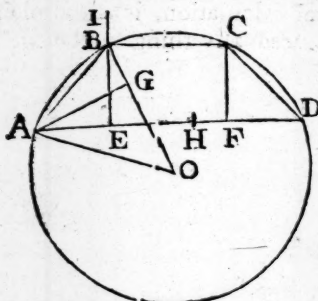
H divide the circumference in three equal parts, and taken KL, in OK produced, equal to $\frac{1}{2}IK$; draw LI, and KB, parallel thereto, meeting OI in B; make OA and OC each = OB; and upon the centers A, B, and C, through A, I, K let the three required circles be described.

PROBLEM XIV.

Supposing AB and AD to be two Arcs of a Circle, in the Ratio of 1 to 3; and that both their Chords AB and AD are given; it is required to find the Radius of the Circle.

Let the arch BD be equally divided in C, and upon AD let fall the perpendiculars BE and CF; also let AG be perpendicular to the radius OB: and put $AB=a$, $AD=b$, and $AO (=OB) = x$.

It is evident, because AB, BC and CD are all equal, that $AE=DF$, and $EF=BC=a$: therefore $AE + DF = b - a$, and $AE = \frac{b-a}{2}$. It also appears that



the triangles ABE and AOG are equiangular, because the angle BAD, standing upon the arch BD, is equal to the angle O, at the center, stand-

ing upon AB ($=\frac{1}{2}BD$): hence we have, $AB (a) :$

$AE \left(\frac{b-a}{2} \right) :: AO (x) : OG = \frac{b-a \times x}{2a}$. But

$AB^2 = OA^2 + OB^2 - 2OB \times OG$; that is, in species,

$a^2 = x^2 + x^2 - \frac{b-a \times x^2}{a}$; or $a^3 = 3ax^2 - bx^2$. Therefore

$$x = \sqrt{\frac{a^3}{3a-b}} = a \sqrt{\frac{a}{3a-b}}.$$

From

From the same equation, if the radius AO (x) and the chord (b) of an arch ABD be supposed given, the chord AB (a) of the sub-triple of that arch, may be determined: but this, by-the-bye.

Geometrically.

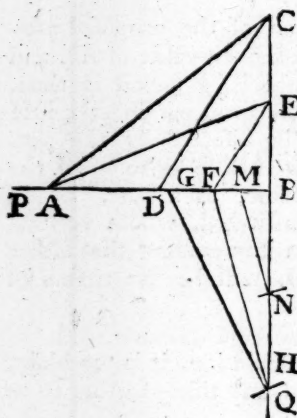
The geometrical construction of the proposed problem is also obvious from the known value of AE and the equality of the angles O and EAB ; and is thus. Draw AD of the given length, from which take $DH = AB$; let the remainder AH be bisected by the perpendicular EI ; to which draw AB so as to be of the given length; and upon the same, as a base, let an isosceles triangle AOB be constituted, whose vertical angle O shall be $= EAB$; then it is evident that either of the equal sides AQ , or BO , will be the radius of the circle.

As to the trigonometrical calculation, it is too plain, from the construction, to need any thing further to be said about it.

PROBLEM

PROBLEM XV.

Having the Lengths of two Lines AE and CD drawn from the acute Angles of a right-angled Triangle ABC, to bisect, and terminate in, the opposite Sides; to determine the Triangle.



Call AE, a ; CD, b ; and BD (or AD) x ; then will $\overline{CB}^2 = b^2 - x^2$; whence $\overline{BE}^2 (= \frac{1}{4}\overline{CB}^2) = \frac{bb - xx}{4}$; and therefore $a^2 (= \overline{AE}^2) = \frac{bb - xx}{4} + 4xx$.

Hence $15xx = 4aa - bb$;

and $x = \sqrt{\frac{4aa - bb}{15}}$.

Geometrically.

If EF be supposed parallel to CD, meeting AB in F, the length thereof, being half that of CD, will consequently be given: whence $\overline{AB}^2 - \overline{BF}^2 (= \overline{AE}^2 - \overline{EF}^2)$ is also given.

Therefore the problem is reduced to this; to determine two lines AB and BF, in the ratio of 4 to 1, so that the difference of their squares may be equal to the difference of the squares of two given lines AE and EF.

Hence, having drawn two indefinite lines BP and BQ at right-angles to each other, take BG, in the former, equal to EF; and from the point G, to BQ, draw

draw $GH=AE$: so shall $\overline{AB}^2 - \overline{BF}^2 (= \overline{AE}^2 - \overline{EF}^2) = \overline{BH}^2$ (*Elem. Cor. 8. 2*). Therefore, if from any point M in PB , to BQ there be drawn $MN=MB$; it is evident that a line, HF , drawn from H parallel to NM , will cut off BF as required.—This problem becomes impossible when either of the two given lines is greater than the double of the other.

PROBLEM XVI.

The Length and Position of a Right-line DE, drawn parallel to the Base of a given right-angled Triangle ABC being known; it is proposed to draw another Right-line CF, from the Vertex of the Triangle, so that the Part thereof (FG) intercepted by AB and ED, may be equal to (EG) the Part of ED intercepted by CA and CF.

Upon AB let fall the perpendicular GH :

and put $ED=a$, $CD=b$,

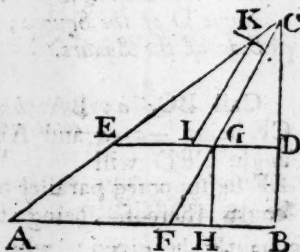
$DB=c$, and $EG=x$:

then, from the similarity of the triangles CDG and

GHF , it will be, $CD (b) :$

$DG (a-x) :: GH (c) :$

$$HF = \frac{c \times a - x}{b} :$$



and therefore $\frac{c^2 \times a - x^2}{b^2} + c^2 (HF^2 + GH^2) = x^2 (GF^2)$

Whence, by reduction, $b^2 x^2 - c^2 x^2 + 2c^2 ax = c^2 a^2 + c^2 b^2$;

$$\text{and } x^2 + \frac{2ac^2}{bb-cc} \times x = \frac{a^2 + b^2 \times c^2}{bb-cc}.$$

$$\text{From which } x \text{ is found } = \frac{bc \sqrt{aa + bb - cc - ac^2}}{bb-cc}.$$

P

Geometrically.

Geometrically.

Since GF is to GC every where in the given ratio of DB to DC, GE, in the required position, must therefore be to GC, in the same, given, ratio. Hence, if, in ED, there be taken EI=DB, and from the center I, at the distance of CD, an arch be described, intersecting EC in K; then a line CGF, drawn parallel to the radius KI, will determine both the length and position of GF: for it is evident that $CG : EG :: IK : EI :: CD : DB :: CG : GF$, and consequently that $EG = GF$.—This problem appears to be impossible when BD is greater than CE.

PROBLEM XVII.

Supposing the Area of a Square BE DF, formed within a given right-angled Triangle ABC, to be equal to the Area of the Triangle ADC, made by drawing Lines from the Extremes of the Hypotenuse to the adjacent Angle D of the Square; it is proposed to determine the Side of the Square.

Call BC, a ; BA, b ; and BE (or BF) x : then, CE being $=a-x$ and AF $=b-x$, the area of the triangle CED will

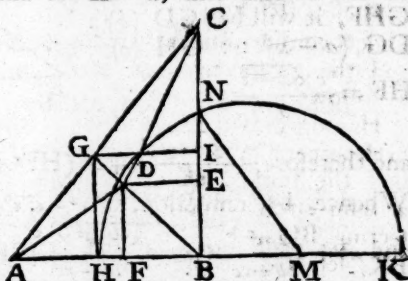
$$\text{be} = \frac{x \times a - x^2}{2},$$

and that of the triangle AFD

$$= \frac{x \times b - x^2}{2}.$$

But it is evident that BEDF + CED + AFD + ADC is $= ABC$: which, because ADC is equal to BEDF, also gives $2BEDF + CED +$

$$AFD = ABC, \text{ that is, } 2x^2 + \frac{x \times a - x^2}{2} + \frac{x \times b - x^2}{2} =$$



$$= \frac{ab}{2} : \text{ hence } x^2 + \frac{a+b}{2} \times x = \frac{ab}{2}; \text{ and } x = \sqrt{\frac{ab}{2} + \frac{1}{4}a + \frac{1}{4}b}^2 - \frac{a+b}{4}. \text{ Q. E. I.}$$

Geometrically.

If BD be produced to meet AC in G, and GH be drawn perpendicular to AB, it is evident that the triangle ADC will be to the triangle ABC ($\frac{AB \times BC}{2}$) as GD : GB, or as HF to HB (*Elem. 8. and 14. of the 4.*): and it also appears (*by Elem. 14. 4. 1st edit.*) that $\frac{HB \times AB+BC}{2}$ is = $AB \times BC$. Therefore it follows that triangle ADC : $\frac{HB \times AB+BC}{2} :: HF : HB :: HF \times \frac{AB+BC}{2} : HB \times \frac{AB+BC}{2}$; and consequently that the triangle ADC = $HF \times \frac{AB+BC}{2} = HF \times BK$; by taking BK (in AB produced) equal to $\frac{AB+BC}{2}$.

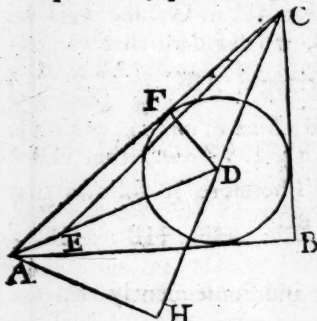
Hence, BF^2 being (=ADC) = $HF \times BK$, the case under consideration is reduced to our 7th problem; and the geometrical construction will, therefore, be as follows.

Having drawn BG (to bisect the angle ABC) and GH perpendicular to AB, and also taken BK equal to half the sum of AB and BC (*as above intimated*), let a semi-circle, upon HK, be next described, intersecting BC in N; from which point, to the middle of BK, let NM be drawn; then make MF=MN, and BF will be the side of the square; as is manifest from the problem above quoted.

PROBLEM XVIII.

Having given the Hypotenuse AC of a right-angled Triangle ABC, and the Difference of two Lines AD and CD, drawn from the Extremes thereof to the Center D of the inscribed Circle; to determine the remaining Sides AB and BC, of the Triangle.

Upon CD, produced, let fall the perpendicular AH; and make $AC=a$, $AD=x$, $DC=y$, and the given difference, $x-y$, $=b$.



It is evident that the angle ADH is $=\angle DAC + \angle DCA = \frac{1}{2} \angle BAC + \frac{1}{2} \angle BCA = \frac{1}{2}$ a right angle; and therefore $AH=HD$

$$= \frac{AD}{\sqrt{2}} = \frac{x}{\sqrt{2}}. \text{ But } CD^2$$

$+AD^2+2DH \times CD=AC^2$; that is, in Species, $y^2+x^2+xy\sqrt{2}=a^2$. Which equation, by substituting $x-b$ instead of y its equal, and denoting $\sqrt{2}$ by c , becomes $x^2-2bx+b^2+x^2+cx^2-cbx=a^2$; that is $2+c \times x^2-2+c \times bx+b^2=a^2$: whence $x^2-bx=\frac{aa-bb}{2+c}$,

$$\text{and } x=\sqrt{\frac{aa-bb}{2+c}}+\frac{1}{2}bb+\frac{b}{2}.$$

Geometrically.

The geometrical construction of this problem, as the angle ADH is given ($=\frac{1}{2}$ a right angle) is exceeding obvious: for, if DE be supposed $=DC$, so that AE may express the given difference of AD and CD, the angle DEC (supposing CE drawn) will be given $=\frac{1}{2}ADH$. Therefore the triangle AEC, by means of the given angle AEC, and the two given sides AE and AC, may be constructed. And then, by producing

producing AE, and making the angle ECD=DEC, the point D will likewise be determined; and consequently the radius of the circle, by letting fall a perpendicular DF, upon AC: whence the circle itself may be described; and two lines may be drawn from A and C to touch the same; and thereby form the triangle ABC, as required.

PROBLEM XIX.

Having the Base AB, the Perpendicular CD, and the Ratio of the two Sides AC, BC, of a Triangle ABC; to find the Sides.

Call AB, a ; CD, b ; and AD, x ; and let the given ratio of AC to BC be expounded by that of m to n . Hence $BD=a-x$; $\overline{AC}^2 (= \overline{CD}^2 + \overline{AD}^2) = bb + xx$ and $\overline{BC}^2 (= \overline{CD}^2 + \overline{BD}^2) = bb + aa - 2ax + xx$:

and therefore
 $mm : nn :: bb + xx : bb + aa - 2ax + xx$.
 From which, by multiplying extremes and means, we have
 $m^2b^2 + m^2a^2 - 2m^2ax + m^2x^2 = n^2b^2 + n^2x^2$:
 whence $mm - nn \times xx - 2mma = nn - mm \times bb - m^2a^2$;
 and $xx - \frac{2mma}{mm - nn} \times x = -bb - \frac{m^2a^2}{mm - nn}$: which, solved,

$$\text{gives } x = \frac{mma}{mm - nn} + \sqrt{\left(\frac{mma}{mm - nn}\right)^2 - b^2}.$$

Geometrically.

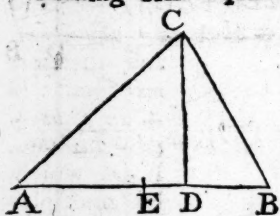
The geometrical construction of this problem is given by *Elem.* 15. 4. 1st edit. For, if the base AB be divided at E in the given ratio of AC to BC, and, in AB produced, there be taken, EO, a fourth proportional

portional to $AE - BE$, AE , and BE , it is there demonstrated, that two lines drawn from A and B to meet any where in the circumference of a circle described through E , from the center O , will be in the same given ratio of AE to BE . Whence it is evident that the intersection of the said circumference with a right-line FG , drawn parallel to AB , at the given distance DC , will determine the vertex of the triangle.

PROBLEM XX.

Having the Base AB , the Perpendicular CD , and the Difference of the two Sides, AC and BC of a Triangle ABC ; to find the Sides.

Making $AE = \frac{1}{2}AB = a$, $CD = b$, $AC - BC = d$, and $ED = x$, we have $AD = a + x$, $BD = a - x$, $AC =$



$= \sqrt{b^2 + a + x}^2$, and $BC = \sqrt{b^2 + a - x}^2$; and consequently $\sqrt{bb + a + x}^2 - d = \sqrt{bb + a - x}^2$. Which equation,

squared, gives $bb + a + x^2 - 2d\sqrt{bb + a + x}^2 + dd = bb + a - x^2$; and this, by reduction, becomes $4ax + dd = 2d\sqrt{bb + a + x}^2$. And this, again squared, produces $16a^2x^2 + 8addx + d^4 = 4dd \times bb + aa + 2ax + xx$; or, $16a^2x^2 + d^4 = 4dd \times aa + bb + 4ddxx$. Whence $x =$

$$\sqrt{\frac{4dd \times aa + bb - d^4}{16aa - 4dd}}$$

The geometrical construction of this problem being only a particular case of a more general one, given at large hereafter (problem 49) I shall not insert it here: but observe, with respect to the algebraical solution, that, if d be supposed to denote the sum, instead of the difference, of the sides, the value of x (or DE) will be

be given by the very equation above exhibited ; as is manifest from the process.

PROBLEM XXI.

Having the Base AB, the Perpendicular CD, and the Rectangle of the two Sides AC and BC, of a Triangle ABC ; to determine the Triangle.

By retaining the notation of the preceding problem, and making the given rectangle $= c^2$, we have

$$\sqrt{bb+ax+x^2} \times \sqrt{bb+a-x}^2 = c^2 ; \text{ and therefore}$$

$$\frac{bb+ax+x^2}{c^2} \times \frac{bb+a-x}{c^2} = \frac{c^4}{c^4} ; \text{ that is } \frac{b^4+b^2 \times 2ax+2x^2}{c^4} + \frac{a^4-2a^2x^2+x^4}{c^4} = \frac{c^4}{c^4} .$$

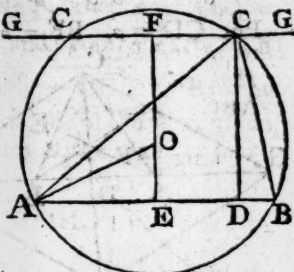
$$\text{Whence } \frac{x^4+2bb-2aa \times x^2}{c^4} = \frac{c^4}{c^4} - \frac{b^4-2a^2b^2-a^4}{c^4} .$$

$$\text{Which, solved, gives } x = \sqrt{aa-bb \pm \sqrt{c^4-4a^2b^2}} .$$

Geometrically.

The magnitude of the rectangle under the two unknown lines, AC and BC, being given, two other lines must, therefore, be assigned, containing an equal rectangle ; whereof one being given, the other will also become known. (*Vid. observation 5, p. 88.*) But it is known that the rectangle under the, given, perpendicular CD and the diameter of a circle circumscribing the triangle, is equal to the rectangle under the said, unknown, sides of the triangle (*Elem. 25. 3*) : hence the diameter of the circumscribing circle is given ; and from thence the following construction.

Find a third-proportional to CD and the side of the square, expressing the magnitude of the proposed rectangle ; and with the half thereof, from the point A (or B) describe an arch, cutting EF, perpendicular to AB,

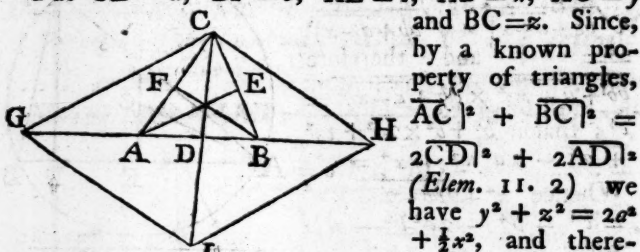


AB, in O; from which point, as a center, with the same radius, let a circle ACB be described; then a right-line, GFG, drawn parallel to AB, at the given distance DC, will intersect the said circle in the vertex of the triangle.

PROBLEM XXII.

The Lengths of three Lines AE, BF and CD, drawn from the Angles to the Middle of the opposite Sides of a Triangle, being given; to find the Sides.

Put $CD = a$, $BF = b$, $AE = c$, $AB = x$, $AC = y$ and $BC = z$. Since,



by a known property of triangles,
 $\overline{AC}^2 + \overline{BC}^2 = 2\overline{CD}^2 + 2\overline{AD}^2$
 (Elem. 11. 2) we
 have $y^2 + z^2 = 2a^2 + \frac{1}{2}x^2$, and there-

fore $y^2 + z^2 - \frac{1}{2}x^2 = 2a^2$. In the same manner

$$x^2 + z^2 - \frac{1}{2}y^2 = 2b^2.$$

$$x^2 + y^2 - \frac{1}{2}z^2 = 2c^2.$$

From whence (by taking the former of these equations from twice the sum of the two latter) there comes out

$$4x^2 + \frac{1}{2}x^2 = 2 \times 2b^2 + 2c^2 - a^2 : \text{ and consequently}$$

$$x = \frac{2}{3} \sqrt{2b^2 + 2c^2 - a^2}. \text{ By the same argument, } y = \frac{2}{3} \sqrt{2a^2 + 2c^2 - b^2}; \text{ and } z = \frac{2}{3} \sqrt{2a^2 + 2b^2 - c^2}.$$

Geometrically.

If CG and CH be drawn parallel to AE and BF, meeting AB, produced, in G and H; it is plain, because $CE = BE$, and $CF = AF$, that $AG = AB = BH$; and also that $CG = 2AE$, and $CH = 2BF$. Therefore, the two sides CG, CH, and the line CD, bisecting the base of the triangle GCH being given, the diagonal CI ($= 2CD$) of the parallelogram GCHI (as well

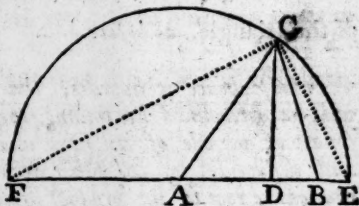
well as the sides) will be given (*Elem.* 12. 2). Hence, in order to the construction, let a triangle CGI, whose three sides are equal to the doubles of the three given lines, be constituted, and draw GDH to bisect CI in D; also set off DA and DB, each equal to $\frac{1}{2}$ of GD; join A, C, and B, C, and the thing is done.

From this construction we have the very same numerical solution as from the algebraic process: for, since, $2GD^2 + 2CD^2 = CG^2 + CH^2$ (*Elem.* 11. 2.) $= 4AE^2 + 4BF^2$, thence is $GD = \sqrt{2AE^2 + 2BF^2 - CD^2}$, and consequently $AB (= \frac{2}{3}GD) = \frac{2}{3} \sqrt{2AE^2 + 2BF^2 - CD^2}$. By the construction it also appears that no one of the three given lines must be greater than the sum of the other two.

PROBLEM XXIII.

All the Sides of a Triangle ABC being given; to find the Perpendicular CD, the Segments of the Base AD and BD, together with the Area of the Triangle.

Put $AC=a$, $AB=b$, $BC=c$, and $AD=x$: then $BD=b-x$; and $c^2 - (b-x)^2 (= CD^2) = a^2 - x^2$; that is, $c^2 - b^2 + 2bx - x^2 = a^2 - x^2$. Whence $2bx = a^2 + b^2 - c^2$, and $x = \frac{aa+bb-cc}{2b}$.



$$\begin{aligned} \text{Now } CD^2 &= AC^2 - AD^2 = AC + AD \times AC - AD \\ &= a + \frac{aa+bb-cc}{2b} \times a - \frac{aa+bb-cc}{2b} = \frac{aa+2ab+bb-cc}{2b} \times \\ &= \frac{-aa+2ab-bb+cc}{2b} = \frac{(b+a)^2 - c^2}{2b} \times \frac{c^2 - (b-a)^2}{2b}. \end{aligned}$$

Q

Hence

Hence $CD = \frac{1}{2b} \sqrt{b+a^2-c^2 \times c^2-b-a^2}$; and
 the area $\left(\frac{CD \times AB}{2} \right) = \frac{1}{4} \sqrt{b+a^2-c^2 \times c^2-b-a^2}$.

Geometrically.

From the center A, with the radius AC, let a semi-circle ECF be described, cutting AB produced in E and F; so that BF may be the sum, and BE the difference of the sides AC and AB: also let EC and FC be drawn.

Then will $\overline{FC}^2 = FE \times FD (= 2AF \times FD)$; and $\overline{EC}^2 = FE \times ED (= 2AE \times ED)$, by *Elem. Corol. to 20. 4.*

Also $\overline{BC}^2 = \overline{BF}^2 + \overline{FC}^2 (= 2AF \times FD) - 2FB \times FD = \overline{BF}^2 - 2AB \times FD$ (*Elem. Cor. 2. to 9. 2.*): and likewise $\overline{BC}^2 = \overline{BE}^2 + \overline{EC}^2 (= 2AE \times ED) - 2EB \times ED = \overline{BE}^2 + 2AB \times ED$.

Hence it appears that $2AB + FD$ is $= \overline{BF}^2 - \overline{BC}^2$; and $2AB \times ED = \overline{BC}^2 - \overline{BE}^2$: and, consequently, that $\overline{BF}^2 - \overline{BC}^2 \times \overline{BC}^2 - \overline{BE}^2 = 4\overline{AB}^2 \times FD \times ED = 4\overline{AB}^2 \times \overline{DC}^2$ (by *Elem. Cor. to 19. 4.*). Therefore $\frac{AB \times DC}{2} = \frac{1}{4} \sqrt{\overline{BF}^2 - \overline{BC}^2 \times \overline{BC}^2 - \overline{BE}^2}$ = the area of the triangle, as before.

From whence it appears, that the area of any triangle will be determined by finding the Differences between the Square of any one of its sides and the Squares of the sum, and difference, of the other two; and then taking $\frac{1}{4}$ of the square root of the product arising by the multiplication of the said differences into each other.

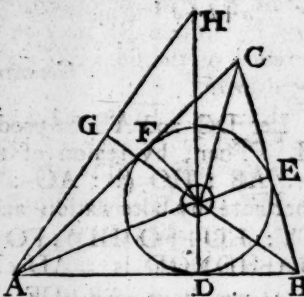
PROBLEM

PROBLEM XXIV.

Having all the Sides of a Triangle ABC; to find the Radius of its inscribed Circle DEF.

From the center O, to the angular points and the points of contact, let lines be drawn; and, upon BO produced, let fall a perpendicular AG.

It is plain, in the first place (because $OD=OE=OF$) that $AD=AF$, $BD=BE$, and $CF=CE$: therefore, by addition, $BD+CF (=BE+CE) = BC$: take each of these equal quantities from $AB+AC$, and there will remain $AD+AF = AB+AC-BC$; from whence (AD being $=AF$) we get AD (or AF) = $\frac{AB+AC-BC}{2}$. By subtracting of which from AB , and from AC , we also have $BD = \frac{AB+BC-AC}{2}$, and $CF = \frac{AC+BC-AB}{2}$.



Moreover, it is evident that the triangles AOG and COF are similar: for the sum of all the angles at the center, $DOE+DOF+FOE$ being $=4$ right angles, the sum of their halves, $BOD+DOA+COF$, must be $=2$ right angles $=BOD+DOA+AOG$; and consequently $COF=AOG$.

Now let the values of AB , BD , and CF (found above) be denoted by a , b and c , respectively; and put OD ($OE=OF$) $=x$: then, it will be BO ($\sqrt{bb+xx}$): OD (x) :: AB ($a+b$): $AG = \frac{ax+bx}{\sqrt{bb+xx}}$; and BO :

Q 2

BD.

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$$BD :: AB : BG = \frac{ab+bb}{\sqrt{bb+xx}}. \text{ Therefore OG (BG-BO) } = \frac{ab+bb}{\sqrt{bb+xx}} - \sqrt{bb+xx} = \frac{ab-xx}{\sqrt{bb+xx}}. \text{ But, AG : OG :: CF : OF; or } ax+bx : ab-xx :: c : x;$$

whence $ax^2+bx^2=abc-cx^2$; and consequently $x = \sqrt{\frac{abc}{a+b+c}}.$

Geometrically.

Let DO and AG be produced to meet each other in H. Then, by reason of the similar triangles, it will be, $AB : HO :: AG : OG :: CF : FO$; and therefore by alternation and composition, $AB+CF : CF :: HO+FO (HD) : FO (OD) :: HD \times OD : OD^2$. But $HD \times OD = AD \times BD$ (*Elem.* 24. 3); therefore we have $AB+CF : CF :: AD \times BD : OD^2 = \frac{AD \times BD \times CF}{AD+BD+CF}$, the very same as before.

From this conclusion, the rule in common practice, for finding the area of a triangle, having the three sides given, is easily deduced: for it is evident that the area of the triangle ABC is equal to the radius (OD) drawn into the half sum of the sides ($AD+BD+BF$); that is $= \sqrt{AD+BD+CF \times AD \times BD \times CF}$. Where AD, BD, and CF, are the differences between the half sum and each particular side.

PROBLEM

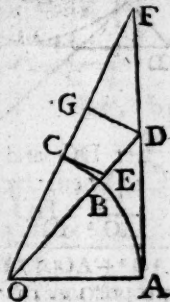
PROBLEM XXV.

The Radius of a Circle and the Tangents of two Arcs thereof being given, to determine the Tangent of the Sum of those Arcs.

Let AB and BC be the proposed arcs, whereof the given tangents are AD and CE; and let the former of these be continued out to meet the radius OC, produced, in F; so shall AF be the tangent of AC, the sum of the said arcs.

Now, calling AO, r ; AD, m ; CE, n ; AF, x ; and FO, y ; and making DG perpendicular to FO; we have (by similar triangles) OF (y): AO

$$(r) :: DF (x-m) : DG = \frac{rx - rm}{y}$$



Also OF (y): AF (x) :: DF ($x-m$): FG = $\frac{xx - mx}{y}$;
 from which last we have OG (= OF - FG) = $y - \frac{xx - mx}{y} = \frac{yy - xx + mx}{y} = \frac{rr + mx}{y}$ (because $yy - xx = rr$.)

But OG ($\frac{rr + mx}{y}$): DG ($\frac{rx - rm}{y}$) :: OC (r): CE (n), and consequently $r^2x - r^2m = r^2n + mn$. Whence
 $x = \frac{rr \times m + n}{rr - mn}$; or AF = $\frac{AO^2 \times AD + CE}{AO^2 - AD \times CE}$.

Otherwise.

Otherwise.

Let AD and AE be the tangents of the two arcs AB and AC, and BF that of their sum BC; also let EG be drawn perpendicular to OD, intersecting the radius OA in H. Then, by reason of the similar triangles, it will be $AO : AD :: AE : AH$. And, $OH (AO - AH) : DE (:: OG : GE) :: OB (AO) : BF = \frac{AO \times DE}{AO - AH} =$

$\frac{AO^2 \times DE}{AO^2 - AO \times AH} = \frac{AO^2 \times AD + AE}{AO^2 - AD \times AE}$; because, by the first proportion $AO \times AH = AD \times AE$. Which conclusion is the very same with that above.

If the tangents of the arcs AC and AB (*fig. 1.*) were to be given, in order to find the tangent of their difference BC; then, by the proportion $\frac{rr + mx}{y} :$

$\frac{rx - rm}{y} :: r : n$ (above derived), we should have $n = \frac{rr \times x - m}{rr + mx}$; or $CE = \frac{AO^2 \times AF - AD}{AO^2 + AF \times AD}$.

PROBLEM XXVI.

The Ratio of the Sines DE, FG of two Arcs AD, AF, of a given Circle, together with that of their Tangents AB, AC being given; to find both the Sines and the Tangents.

Put the radius $AO = a$; and let the given ratio of AB to AC be that of m to n ; moreover let DE be to FG

FG as p to q ; and call AB, x ; and AC, y : then
(by similar triangles) $\overline{OB}^2 : (a^2 + x^2) : \overline{AB}^2 : (x^2) ::$

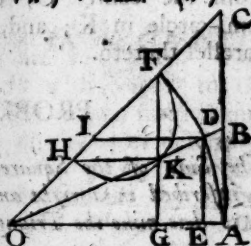
$$\overline{OD}^2 : (a^2) : \overline{DE}^2 : \frac{a^2 x^2}{aa + xx};$$

$$\text{and } \overline{OC}^2 : (a^2 + y^2) : \overline{AC}^2 : (y^2)$$

$$:: \overline{OF}^2 : (a^2) : \overline{FG}^2 : \frac{a^2 y^2}{aa + yy}.$$

Therefore, by the question, $p^2 :$

$$q^2 :: \frac{a^2 x^2}{aa + xx} : \frac{a^2 y^2}{aa + yy}, \text{ and}$$



consequently $p^2 y^2 \times \overline{a^2 + x^2} = q^2 x^2 \times \overline{a^2 + y^2}$. But, by

the question, we also have, $m : n :: x : y, = \frac{nx}{m}$;

which value, substituted in the preceding equation,

gives $\frac{n^2 p^2 x^2}{mm} \times \overline{a^2 + x^2} = q^2 x^2 \times \overline{a^2 + \frac{nnxx}{mm}}$: whence

$$n^2 p^2 \times \overline{a^2 + x^2} = q^2 \times \overline{a^2 m^2 + n^2 x^2}; \text{ and } x = \frac{a}{n} \times$$

$$\sqrt{\frac{nnpp - mmqq}{qq - pp}}.$$

From which AC, DE and FG

are also given.

Geometrically.

Since the ratio of AB to AC is given, as m to n ; GK (supposing K to be the intersection of FG and OB) will be to GF, in the same given ratio: and, if (agreeable to the 3^d general observation) KH be drawn parallel to AO, meeting OF in H, OH will be to OF still in the same given ratio.

Again, if DI be drawn parallel to AO, meeting OF in I; then OK : OH ($::$ OD (OF) : OI $::$ FG : DE) $::$ $q : p$; whence OK is also given; and from thence the following construction.

In any radius OF of the given circle, take OH to OF in the given ratio of m to n ; and upon HF let a semi-circle be described: take also a fourth proportional

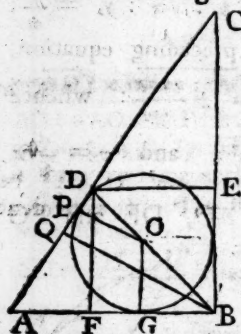
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portional to p , q , and OH ; with which, as a radius, from the center O , describe an arch, cutting the semi-circle in K ; and, having drawn HK , make OA parallel thereto.

PROBLEM XXVII.

The Side of the Square, and the Radius of the Circle, inscribed in a right angled Triangle ABC, being given; to determine the Triangle.

Draw the diagonal BD of the square; and from the center O of the given circle, to the points of contact,



draw the radii OG and OP ; and upon the hypotenuse, AC let fall the perpendicular BQ ; calling the side of the square, a ; the radius of the circle, b ; and AQ , x : then because of the parallel lines, we shall have $FG (a-b) : BF (a) :: (OD : BD ::) OP (b) : BQ = \frac{ab}{a-b}$:

whence $DQ (= \sqrt{BD^2 - BQ^2})$

$$= \sqrt{2a^2 - \frac{aabb}{a-b}^2}, \text{ is also given.}$$

Let it, for brevity-sake, be denoted by c ; and let $BQ (\frac{ab}{a-b}) = d$: then we shall have, $AQ (x) : BQ (d) :: BQ (d) : CQ = \frac{dd}{x}$; and also (by *Elem.* 18. 4.) $AD (x+c) :$

$CD (\frac{dd}{x} - c) :: AB : BC :: AQ (x) : BQ (d)$; whence, by multiplying extremes and means, we get $dx + cd = dd - cx$; and, from thence, $x = \frac{d \times d - c}{d + c}$. By means whereof every thing else is readily found.

Geometrically.

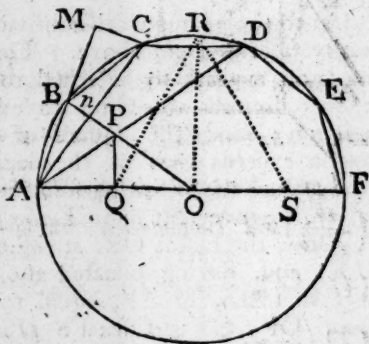
Geometrically.

The geometrical construction, and the trigonometrical solution of this problem are very easy. For, since the position of the point D with respect to the circle, is given; a right-line DPA drawn from thence (by *Elem.* 21. 5.) to touch the circle, will determine the triangle: and then, from the given lines OP and OD, the angle D may be found: by means whereof and ABD ($=\frac{1}{2}$ a right angle) together with the given side BD, all the rest will become known.

PROBLEM XXVIII.

To determine the Sides of a regular Pentagon and Decagon, inscribed in a given Circle.

Let AB, BC, CD, &c. be sides of the decagon, and AC a side of the pentagon: and let AD be drawn, intersecting the radius OB in P. It is evident in the first place, that the angles BAP and OAP, standing on the equal arches BD and BF, are equal to one another, and also equal, each of them, to the angle AOP, insisting on the arch AB (*Elem.* 10. 3.) And secondly, that the triangle BAP (as well as APO) is an isosceles one, because the perpendicular AnC, makes equal angles BAC, DAC with the two sides AB, AP of the triangle. Hence it appears very plain that all the three lines AB, AP, and OP are equal among themselves; and likewise that $AO (OB) : AB (OP) :: OP (AB) : BP$ (by *Elem.* 18. 4.) seeing the angle BAO



is bisected by AP. Moreover, by letting fall the two perpendiculars PQ and CM, upon AO and ABM, the triangles BCM and APQ (as BC is = AB = AP, and the angle MBC = MAD = PAQ) will appear to be equal in all respects; and so, BM being (=AQ) = $\frac{1}{2}$ AO, we have $\overline{AC}^2$ (= $\overline{BC}^2 + \overline{AB}^2 + 2BM \times AB$, *Elem.* 11. 2.) = $\overline{BC}^2 + \overline{AB}^2 + AO \times AB$. But, by the above proportion, $\overline{AB}^2$ is = $AO \times BP$: therefore, by writing $AO \times BP$ in the room of $\overline{AB}^2$, we get $\overline{AC}^2$ = $\overline{BC}^2 + AO \times AB + AO \times BP$ = $\overline{BC}^2 + \overline{AO}^2$: whence (BC being first found) AC will also become known.—If AO be now denoted by a , and AB by x ; then from the equality of $\overline{AB}^2$ and $AO \times BP$, you will have $x^2 = a \times a - x$; from which x will be found = $\sqrt{\frac{5aa}{4} - \frac{a}{2}}$; and from thence AC (= $\sqrt{xx + aa}$) = $a\sqrt{\frac{5 - \sqrt{5}}{2}}$.

As to the geometrical construction, it likewise very easily follows from above. For, since the side of the decagon appears to be equal to the greater part of the radius divided according to extreme and mean proportion; and the square of the side of the pentagon exceeds that of the decagon by the square of the radius; the following solution (given by Dr. Barrow, as an improvement upon *Euclid's*) is manifest.

Draw the radius OR, at right-angles to the diameter AF; and, having bisected the radius AO in Q, let QS be taken, in AF, equal to the distance QR: so shall OF : OS :: OS : FS (*Elem.* 19. 5.) and consequently OS = AB the side of the decagon.

And, because $\overline{RS}^2$ (supposing RS drawn) is = $\overline{OS}^2 + \overline{OR}^2$, it is plain also that RS will be equal to the side AC of the pentagon.

PROBLEM

PROBLEM XXIX.

Having the Hypotenuse AC of a right-angled Triangle ABC, and also the Radius of the inscribed Circle DEFG; to find the two Legs AB and BC.

From the center D of the given circle, to the points of contact, let DE, DF, and DG be drawn; also draw AD and CD; and put DE (= DG = DF) = a , AC = b , AB = x , and BC = y .

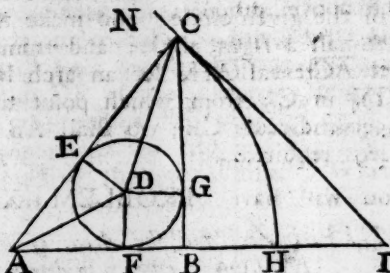
It is evident, that CE must be = CG = $y - a$; because the right-angled triangles CDE and CDG, having DE = DG, and CD common, are equal in all respects. In the very same manner is AE = AF = $x - a$.

Therefore $y - a + x - a = b$ (=AC); from which equation we have $x + y = b + 2a$. But, from the property of right-angled triangles, we also have $xx + yy = bb$. And, if from the double of this, the square of the former equation be subtracted, there will remain $xx - 2xy + yy = bb - 4ab - 4aa$.

From whence, by extracting the square root, on both sides, we get $x - y = \sqrt{bb - 4ab - 4aa}$. Which last equation, added to, and subtracted from, the first, gives $2x = 2a + b + \sqrt{bb - 4ab - 4aa}$, and $2y = 2a + b - \sqrt{bb - 4ab - 4aa}$.

Geometrically.

Seeing the difference between each leg of the triangle and the adjacent segment of the hypotenuse, is equal to the radius of the circle, it is plain that the sum of the two legs (AB + BC) will exceed the sum



of the two segments (or the whole hypotenuse) by twice the radius; and will therefore be given $= AC + 2BF$. Whence, if BI be supposed equal to BC , AI will likewise be given $= AC + 2BF$, and the angle $BIC =$ half a right-one. Hence the following construction.

Draw an indefinite line, in which take AH equal to the given hypotenuse, and HI equal to the diameter of the given circle; also make the angle AIN equal to half a right angle; and from the center A , with the interval AH , let an arch be described, meeting IN in C ; from which point upon AP let fall the perpendicular CB ; so shall AB and BC be the two legs required.

PROBLEM XXX.

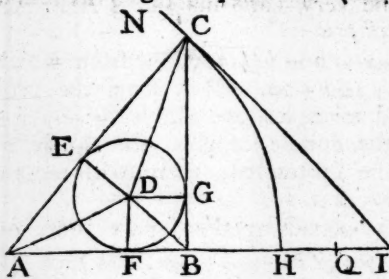
The Perimeter, and the Area, of a right-angled Triangle ABC being given; to determine the Triangle.

Put the given perimeter $(AB + BC + AC) = p$, the area $(\frac{1}{2} AB \times BC) = a^2$; and let half the sum of the legs AB and BC be denoted by x , and half their difference by y ; then, AB being $= x + y$, $BC = x - y$, and $AC = p - 2x$, we shall have

$x + y \times x - y (= AB \times BC) = 2a^2$; and $x + y)^2 + x - y)^2 (= AC^2) = p - 2x)^2$; that is, by reduction, $xx - yy = 2a^2$; and $2xx + 2yy = pp - 4px + 4xx$.

Now, by the addition of the latter of these equations to the double of the former, there arises $4xx = pp - 4px + 4xx + 4a^2$: whence x comes out $= \frac{pp + 4aa}{4p} = \frac{1}{4}p + \frac{aa}{p}$. From which value that of $y = \sqrt{x^2 - 2a^2}$, will likewise be given.

Geometrically.



Geometrically.

If from the center D , of a circle inscribed in the triangle, lines be supposed drawn to the angular points, the proposed triangle will, by that means, be divided into three others ADB , BDC , and ADC ; whose bases are the three sides of the first triangle, and their perpendiculars all radii of the said circle. From which it is evident that the triangle ABC is equal to a rectangle under half the sum of its three sides (which we will here express by the given line AQ) and the radius DF of the inscribed circle; and consequently that the radius DF will be given by taking a third proportional to AQ and the side (a) of the square expressing the given area. Whence, making QH and QI , each, equal to DF , so found; it will appear from the preceding problem that AH will be = the hypotenuse, and AI = to the sum of the two legs, of the proposed triangle: which quantities being both given, the method of construction is manifest from the last problem.

PROBLEM

PROBLEM XXXI.

The Sum, or Difference of the two Legs AC and BC of a right-angled Triangle ABC being given, together with the Sum, or Difference of the Hypotenuse AB and a Perpendicular CD falling thereon from the Right-angle; to find all the Sides of the Triangle.

Put $AC + BC = s$, $AC - BC = d$, $AB + CD = p$,

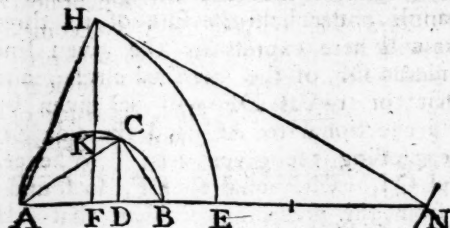
$AB - CD = q$;

then will AC

$$= \frac{s+d}{2}, \quad BC =$$

$$\frac{s-d}{2}, \quad AB =$$

$$\frac{p+q}{2}, \quad CD =$$



$\frac{p-q}{2}$: but $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$; and $AB \times CD (= 2 \text{ area } ABC) = AC \times BC$: which, in species, give $p^2 + 2pq + q^2 = 2s^2 + 2d^2$, and $p^2 - q^2 = s^2 - d^2$.

By adding, and subtracting the double of the last of these equations to and from the former, we have these two other equations,

$$\text{viz. } 3p^2 + 2pq - q^2 = 4s^2,$$

$$\text{and, } -p^2 + 2pq + 3q^2 = 4d^2.$$

From which, when any two of the quantities, s , d , p , q , are given, the other two will, easily, be determined.

Thus, let s and p be given; then, from the former equation, we have $q^2 - 2pq = 3p^2 - 4s^2$; whence $q^2 - 2pq + p^2 = 4p^2 - 4s^2$, and $p - q = 2\sqrt{p^2 - s^2}$: therefore $CD (= \frac{p-q}{2}) = \sqrt{p^2 - s^2}$, and $AB = p - \sqrt{p^2 - s^2}$.

If d and q be given; we shall have $CD = \sqrt{q^2 - d^2}$; because $p^2 - q^2 = s^2 - d^2$, or $p^2 - s^2 = q^2 - d^2$ (p. above.)

But,

But, if s and q be given, then will $3p^2 + 2pq = 4s^2 + q^2$; which, solved, gives $p = \frac{2}{3}\sqrt{3ss + qq} - \frac{1}{3}q$.

Lastly, if d and p be given, we shall have $3q^2 + 2pq = 4d^2 + p^2$, and consequently $q = \frac{2}{3}\sqrt{3dd + pp} - \frac{1}{3}p$.

Geometrically.

The very same properties whereby the algebraical solution is above brought out, lead us also to geometrical constructions of the several cases of the problem under consideration. But it will be sufficient, here, to exhibit that of the case, wherein the sum of the legs (AE) and the difference of the hypotenuse and perpendicular (AF) are supposed given, being the most difficult.

Thus, because $\overline{AC}^2 + \overline{BC}^2 = \overline{AB}^2$,
and $2AC \times BC (= 2AB \times CD) = 2AB \times BF$;
we shall, by adding these equal quantities,
have $\overline{AC}^2 + \overline{BC}^2 + 2AC \times BC$ (or $\overline{AE}^2$, *Elem.* 6. 2.)
 $= \overline{AB}^2 + 2AB \times BF = \overline{AB}^2 + 2AB \times \overline{AB} - \overline{AF}^2 = 3\overline{AB}^2 - 2AB \times AF$.

Therefore, if an arch, from the center A, with the radius AE, be described; and, from its intersection (H) with FH perpendicular to AE, another arch be also described, with the radius 2AE, cutting AE produced in N; then the hypotenuse AB of the required triangle, will be $\frac{1}{3}$ of the line AN thus determined.

For $\overline{HN}^2$ ($4\overline{AE}^2$) being $= \overline{AN}^2 + \overline{AH}^2$ ($\overline{AE}^2$)
 $- 2AN \times AF$ (*Elem.* 10. 2.) and therefore $3\overline{AE}^2 = \overline{AN}^2$
 $- 2AN \times AF$; it is plain, if AB be taken $= \frac{1}{3}AN$,
that $3\overline{AE}^2 = 3\overline{AB} \times 3\overline{AB} - 6AB \times AF$: and consequently
that $\overline{AE}^2 = 3\overline{AB}^2 - 2AB \times AF$, *the very same as above.*

From the value of AB, thus given, what yet remains to be done, will be effected with great facility. For, if in FH there be taken $FK = FB$, and KC be drawn

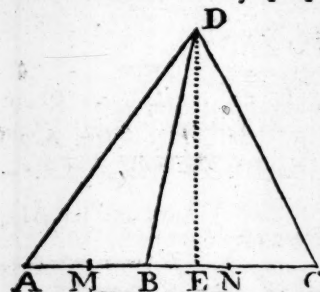
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drawn parallel to AN, intersecting a semi-circle, described upon AE, in the point C, that point will, it is evident, be the vertex of the triangle.

LEMMA.

If a Line be drawn from the Vertex to any Point in the Base of a Triangle, the Sum of the two Solids under the Squares of the two Sides and the alternate Segments of the Base, will be equal to the Solid under the whole Base and its two Segments, together with the Solid under the same Base and the Square of the dividing Line.

Let ACD be any proposed triangle, and BD the dividing line; then, I say, that $\overline{AD}^2 \times BC + \overline{CD}^2 \times AB = AC \times AB \times BC + AC \times \overline{BD}^2$.



For, if AB and CB be bisected in M and N, and a perpendicular DE be let fall upon AC, it is known,

$$\text{that } \begin{cases} \overline{AD}^2 - \overline{BD}^2 = AB \times 2ME \\ \overline{CD}^2 - \overline{BD}^2 = BC \times 2NE \end{cases} \begin{matrix} \text{Elem. 9. of 2.} \\ \text{Cor. 2.} \end{matrix}$$

Whence it follows,

$$\text{that } \overline{AD}^2 \times BC - \overline{BD}^2 \times BC = AB \times BC \times 2ME; \\ \text{and } \overline{CD}^2 \times AB - \overline{BD}^2 \times AB = AB \times BC \times 2NE.$$

Let these equal quantities be added together, and the sums will also be equal;

$$\text{that is, } \overline{AD}^2 \times BC + \overline{CD}^2 \times AB - AC \times \overline{BD}^2 (= \\ AB \times BC \times 2MN) = AB \times BC \times AC; \text{ and consequently} \\ \overline{AD}^2 \times BC + \overline{CD}^2 \times AB = AC \times AB \times BC + AC \times \overline{BD}^2. \\ \text{Q. E. D.}$$

COROL. 1. Hence, if $AB=BC$, then will $\overline{AD}^2 + \overline{CD}^2 = 2AB^2 + 2\overline{BD}^2$.

COROL.

COROL. 2. But, if $AD=DC$, then we shall have
 $\overline{AD}^2 \times BC + \overline{CD}^2 \times AB = \overline{AD}^2 \times BC + \overline{AD}^2 \times$
 $AB = \overline{AD}^2 \times AC$: whence $\overline{AD}^2 \times AC = AB \times BC$
 $\times AC + \overline{BD}^2 \times AC$; and consequently $\overline{AD}^2 = AB \times$
 $BC + \overline{BD}^2$.

COROL. 3. Lastly, if the angle $ADB =$ the angle
 CDB (or $AD : CD :: AB : BC$, *Elem.* 18. 4.) then
 $AD \times BC$ being $= CD \times AB$, it follows that $\overline{AD}^2 \times BC$
 $= AD \times CD \times AB$; and that $\overline{CD}^2 \times AB = AD \times$
 $CD \times BC$. Hence $\overline{AD}^2 \times BC + \overline{CD}^2 \times AB = AD$
 $\times CD \times \overline{AB+BC} = AD \times CD \times AC = AB \times BC \times AC +$
 $\overline{BD}^2 \times AC$ (*p.* above); and consequently $AD \times CD =$
 $AB \times BC + \overline{BD}^2$.

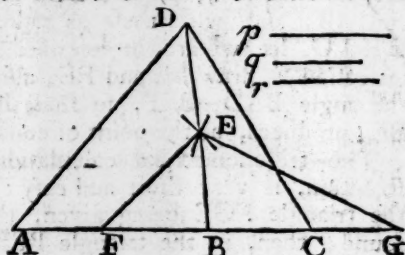
PROBLEM XXXII.

From three given Points, A, B, C, in the same Right-
 line, to draw as many Lines, to meet in a fourth Point
 D, so as to obtain a given Ratio among themselves.

Call AB, a ; BC, b ; AC, c ; and AD, x ; and let AD,
 BD, and CD be,
 in proportion to
 one another, as p ,
 q , and r , respec-
 tively. Then, BD

being $= \frac{qx}{p}$, and

$CD = \frac{rx}{p}$, we



shall, by the preceding lemma, have $x^2 \times b + \frac{r^2 x^2}{pp} \times a =$

$abc + c \times \frac{q^2 x^2}{pp}$.

Whence $bp^3 + ar^3 - cq^2 \times x^2 = abcp^2$;

S

and

and consequently $x = p \sqrt{\frac{abc}{b^2p + ar^2 - cq^2}}$.

After the same manner the problem may be resolved, when, instead of the ratio, the sums, the differences, or the rectangles of the three lines, are given.

Geometrically.

If from any point E, in BD, two lines EF and EG be supposed drawn, so as to form the angles BEF and BEG, respectively, equal to BAD and BCD, it is evident, from the similarity of the triangles BEF, BDA, and BEG, BDC,

that $\begin{cases} BE : BF :: BA : BD \\ BE : BG :: BC : BD. \end{cases}$

And, consequently, that $BF : BG :: BC : BA$. Therefore, if BF be taken $= BC$, BG will be $= AB$.

Moreover, from the abovementioned, similar, triangles,

we have $\begin{cases} BD : AD (:: q : p) :: BF (BC) : FE \\ BD : CD (:: q : r) :: BG (AB) : GE. \end{cases}$

Whence FE and GE are given; and from thence the following construction.

Take $BF = BC$ and $BG = AB$; also take a fourth proportional to q , p , and BC , and another to q , r , and AB ; and, with these as radii, from the centers F and G , let two arcs be described; and, to their intersection E , draw BE and FE ; also draw AD , making the angle $BAD = BEF$, so shall its intersection with BE , produced, be the point of concurrence required.

The trigonometrical calculation, from this construction, is very short and easy: for, all the sides of the triangle FGE being given, the angle F may be found; then, in the triangle BFE , two sides and the included angle being known, every thing else is readily determined.—It may be observed that there is another construction of this problem; by means of the intersection of two circles, so described that lines drawn from the given points to meet in the peripheries thereof, may obtain the given ratios (*see Elem. 15. 4. 1st edit.*) but the method given above I look upon as preferable.—

As

As to the limitations, it is plain the problem becomes impossible when the two circles, described from C and F, do not meet each other; that is, when q is given either, less than the difference, or greater than the sum of $\frac{pb}{c}$ and $\frac{ra}{c}$.

PROBLEM XXXIII.

The two Sides AD, CD, of a Triangle being given in Length, together with the Length of a Line DB dividing the Base AC in a given Ratio; to determine the Base, or Line so divided.

Call AD, a ; BD, b ; CD, c ; and AB, x ; and let the given ratio of AB to BC be that of m to n . Hence

$$BC = \frac{nx}{m}, \text{ and } AC (=x)$$

$$+ \frac{nx}{m} = \frac{m+n \times x}{m}. \text{ And}$$

therefore, by the lemma,

$$a^2 \times \frac{nx}{m} + c^2 \times x = \frac{m+n \times x}{m}$$

$$\times \frac{nx}{m} \times x + \frac{m+n \times x}{m} \times b^2.$$

$$\text{Whence, by reduction, } mna^2 + m^2c^2 - m \times m + n \times b^2 = m + n \times nx^2.$$

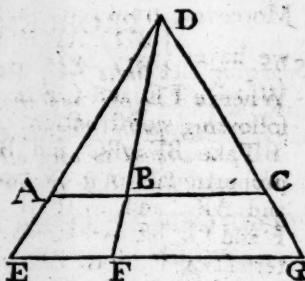
$$\text{And } x = \sqrt{\frac{mna^2 + mmcc - mbb}{m+n \times n} - \frac{mbb}{n}}.$$

If DB be supposed to bisect the base; then, m and n being equal, x becomes $= \sqrt{\frac{aa + cc}{2} - bb}.$

But, if DB be supposed to bisect the vertical angle, we shall have $m : n (:: AD : CD) :: a : c$ (*Elem.* 18. 4.)

S 2

Whence,



Whence, by writing a and c instead of m and n , our equation becomes $x = \sqrt{\frac{a^2c + a^2c^2 - ab^2}{a + c \times c} - \frac{ab^2}{c}} =$

$$\sqrt{a^2 - \frac{ab^2}{c}}.$$

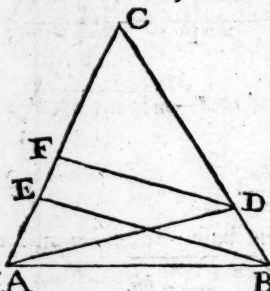
Geometrically.

In any right-line EG, at pleasure, let there be taken EF and FG in the given ratio of AB to BC; and, from the points E, F and G, let three lines be drawn to meet in a point D (by the last problem) so as to obtain the given ratio of AD, BD, and CD, respectively: then, if DA be taken of the given length, and ABC be drawn parallel to EFG, it is manifest that ADC will be the triangle required.

PROBLEM XXXIV.

Supposing two Sides AC, BC, of a Triangle ABC, to be given in Length; and that two Lines BE, AD, drawn from the opposite Angles, to cut off given Segments AE, BD, are equal to each other; it is proposed to determine the other Side AB of the Triangle.

Put $CE=a$, $AE=b$, $CD=c$, $BD=d$, $CA=f$, $CB=g$, $AB=x$, and AD (or BE) $=z$. Then, by the lemma, we have the two following equations.



$g^2 \times b + x^2 \times a = fab + f \times z^2$,
 $f^2 \times d + x^2 \times c = gcd + g \times z^2$.
 From the former of which, multiplied by g , let the latter, multiplied by f , be subtracted, and there will arise $bg^3 - df^3$

$$+ agx^2 - cf x^2 = abfg - cdfg.$$

$$\text{Therefore } x = \sqrt{\frac{ab - cd \times fg + df^3 - bg^3}{ag - cf}}.$$

Geometrically.

Geometrically.

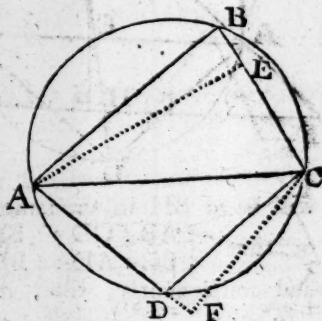
If DF be drawn parallel to BE, the ratio thereof to BE (or AD) will be given, as CD to CB; and CF will likewise be to CE in the same given ratio, and therefore will be given in length. Hence it is evident that the position of the point D with respect to the side AC (first laid down) will be determined by the intersection of two circles; one described from the center C, with the given interval CD; and the other, by *Elem. 15. 4. 1st edit.* so that lines drawn from F and A, to meet any where in the periphery thereof, may obtain the said given ratio of CD to CB. Through which point, so determined, the other side CB of the triangle must be drawn; and then, AB being joined, the thing is done.

PROBLEM XXXV.

All the Sides of a Trapezium ABCD, about which a Circle may be described, being given in length; to determine the Diameter of the Circle.

Let the diagonal AC be drawn, and upon BC and AD let fall the perpendiculars AE and CF. And put $AB=a$, $BC=b$, $CD=c$, $AD=d$, and $BE=x$.

Now the external angle CDF being equal to the internal, opposite, angle B (*Elem. 17. 3.*) the triangles ABE and CDF are similar: and therefore $AB(a) : BE(x)$



$$:: CD(c) : DF = \frac{cx}{a}. \text{ But } \overline{AB}^2 + \overline{BC}^2 - 2BC \times BE$$

(=

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$(=AC)^2 = AD^2 + CD^2 + 2AD \times DF$; that is, in species, $aa+bb-2bx=cc+dd+\frac{2cdx}{a}$.

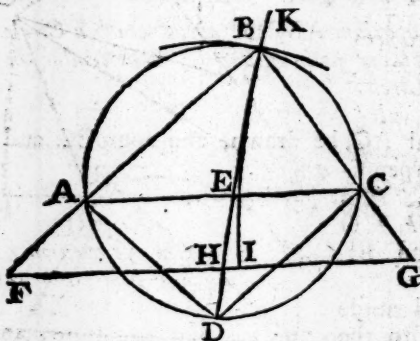
From which equation x is found $=\frac{aa+bb-cc-dd}{2b+\frac{2cd}{a}}$:

whence $AE (= \sqrt{aa-xx})$ and $AC (= \sqrt{aa+bb-2bx})$ will also be given: and then it will be $AE : AC :: AB : \text{the diameter sought (Elem. 24. 3.)}$.

Geometrically.

If the two diagonals of the trapezium be drawn, intersecting each other in E , the triangles ABE and CED , as well as CEB and AED , will be equiangular (Elem. 12. 3.)

Whence $\begin{cases} AB : CD :: BE : CE \\ BC : AD :: BE : AE. \end{cases}$



And, since the ratios of CE and AE to BE are thus given, it follows, that, if any line FG be drawn parallel to AC , the parts thereof GH and FH , intercepted by BC , BE , and BA (produced)

will be to BH in the same given ratios,

or that $\begin{cases} AB : CD :: BH : GH \\ BC : AD :: BH : FH; \end{cases}$

and consequently, that, if BH be taken $=AB$, GH will be given $=CD$, and $FH = \frac{AD \times AB}{BC}$. Whence

the following construction.

Make

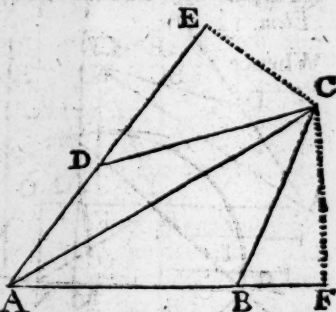
Make FH a fourth-proportional to BC, AB, and AD (by *Elem.* 13. 5); and, in the same line produced, take $HG=CD$: then (by *Elem.* 15. 4. 1st edit.) let a circle IK be described, so that two lines drawn from F and G, to meet any where in the periphery thereof, may obtain the given ratio of AB to CB. And from H, as a center, with the radius AB, let another arch be described, intersecting the former in B; draw BF and BG, in which set off BA and BC of the given lengths, then, through the three points A, B, and C, let a circle be described, and the thing is done.

PROBLEM XXXVI.

All the Sides, and the Area of a Trapezium ABCD being given; to determine the Trapezium.

Suppose the diagonal AC to be drawn; suppose also CE and CF to be perpendicular to AD and AB: and make AD = a , DC = b , BC = c , AB = d , the given area ABCD = r^2 , DE = x , and BF = y .

Then will $aa + bb + 2ax (=AC^2) = cc + dd + 2dy$ (*Elem. Cor. 2. 9. 2.*) and $a\sqrt{bb - xx} + d$



$\sqrt{cc - yy} (=2ADC + 2ABC) = 2r^2$ (by the question.)

Put $c^2 + d^2 - a^2 - b^2 = 2f$; and, by the first equation, you will have $ax - dy = f$.

Moreover, by squaring the two last equations, and then adding them together, you will have

$$a^2b^2 + c^2d^2 + 2ad\sqrt{bb - xx} \times \sqrt{cc - yy} - 2adxy = 4r^4 + f^2.$$

Which, by dividing by $2ad$, and making $g = \frac{4r^4 + f^2}{2ad} - \frac{ab^2}{2d} - \frac{c^2d}{2a}$, is reduced to $\sqrt{bb - xx} \times \sqrt{cc - yy} = g + xy$.

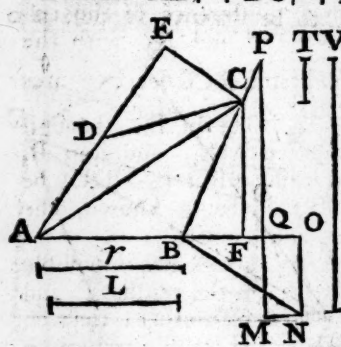
This, squared, gives $b^2c^2 - b^2y^2 - c^2x^2 = g^2 + 2gxy$: which

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which, by writing $\frac{dy+f}{a}$ in the room of y , its equal,
 x , will become $b^2c^2 - b^2y^2 - \frac{c^2 \times dy+f}{aa} = g^2 + 2gy \times$
 $\frac{dy+f}{a}$: whence, putting $b^2c^2 - \frac{c^2f^2}{a^2} - g^2 = b$, $\frac{2c^2df}{a^2}$
 $+\frac{2fg}{a} = k$, and $b^2 + \frac{c^2d^2}{a^2} + \frac{2dg}{a} = l$, we get $b = ky + ly^2$;
 and therefore $y = \sqrt{\frac{b}{l} + \frac{kk}{4ll} - \frac{k}{2l}}$.

Geometrically.

Because $\overline{AD}^2 + \overline{DC}^2 + 2AD \times DE (= \overline{AC}^2) =$
 $\overline{AB}^2 + \overline{BC}^2 + 2AB \times$
 BF (*Elem. Cor. 2. 9. 2.*)
 it is plain that $2AD \times DE$
 $- 2AB \times BF$ is given =
 $\overline{AB}^2 + \overline{BC}^2 - \overline{AD}^2 -$
 $\overline{CD}^2$.



Find (by *Elem. 2. and*
 3. 6.) a line L whose
 square shall be $= \overline{AB}^2$
 $+ \overline{BC}^2 - \overline{AD}^2 - \overline{CD}^2$;
 so shall $2AD \times DE =$
 $2AB \times BF = L^2$, and consequently $DE = \frac{AB \times BF}{AD} =$
 $\frac{L^2}{2AD}$.

But $\frac{AB \times BF}{AD}$, (supposing PQ perpendicular to AB ,
 and BP a fourth proportional to AD , AB , and BC)
 appears to be $= BQ$; and $AB \times CF$ (since $AD : AB$
 $:: BC : BP :: CF : PQ$) will also be equal to $AD \times$
 PQ (*Elem. 10. 4.*)

Hence

Hence we have $DE - BQ = \frac{L^2}{2AD}$, and $AD \times EC + AB \times CF (=2r^2) = AD \times EC + AD \times PQ$: whence, by taking a third proportional (T) to $2AD$, and L , and another (V) to $\frac{1}{2}AD$ and r , there results $DE - BQ = T$; and $EC + PQ = V$.

Therefore the problem is reduced to this; to find two angles CDE , PBQ , so that the sum of their sines CE , PQ , and the Difference of their co-sines DE , BQ (answering to given, but unequal, radii DC , BP) may be both given quantities.

And the construction thereof, (which is exceeding evident) will be as follows.

Draw two lines cutting each other at right-angles in M , in which take $MP = V$, the given sum of the sines; and $MN = T$, the given difference of the co-sines: then, from the centers P and N , with the given radii $PB \left(= \frac{AB \times BC}{AD} \right)$ and DC , let two arcs be described, intersecting each other in B ; through which point draw OA parallel to MN ; and join B , P , and B , N ; so shall PBO and $NBO (=CDE)$ be the two angles required. Which being known, the trapezium itself is very easily constructed.

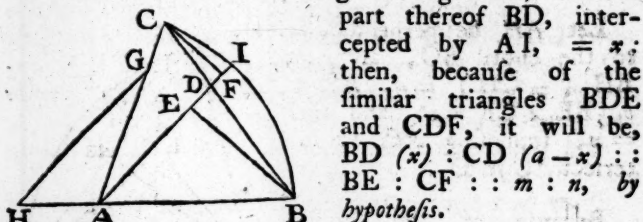
This problem, it may be observed, becomes impossible when the two arcs, described from the centers P and N , do not meet, but fall short of each other; that is, when the given area is greater than that of a trapezium of the same given sides, inscribed in a circle, determined by the preceding problem.

But, besides the above, there is another limit, for the least value of the area (except in one particular case) and the problem will be impossible when one of the two circles falls wholly within the other, as well as when it falls wholly without it: but this last depends upon the particular order of joining the given lines; whereas the case is otherwise with respect to the first, or greatest limit.

PROBLEM XXXVII.

To divide a given Angle BAC into two Parts BAI and CAI, so that their Sines BE and CF may obtain a given ratio; suppose that of m to n .

Put the chord BC of the given angle $=a$, and the part thereof BD, intercepted by AI, $=x$:



then, because of the similar triangles BDE and CDF, it will be, $BD (x) : CD (a-x) :: BE : CF :: m : n$, by hypothesis.

Therefore $nx = m \times a - x$; and consequently $x = \frac{ma}{m+n}$.

From whence, and the given angle ABD, the angle BAD will be found.

The geometrical construction (like the algebraical operation) may be performed by dividing the subtense BC in the given ratio of m to n : but the following method is preferable.

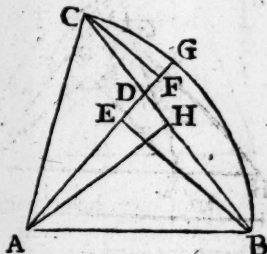
In AC take $AG = m$; and, in BA produced, take $AH = n$; then a line AE drawn parallel to that joining the points G and H, will divide the angle as required. For, by trigonometry, $AG (m) : AH (n) :: \text{fine } AHG (BAE) : \text{fine } AGH (CAE)$: from this construction the numerical solution is exceeding easy; both the sides AG and AH, and the included angle HAG being given.

PROBLEM

PROBLEM XXXVIII.

To divide a given Angle BAC into two such Parts BAG, CAG, that the Rectangle under their Sines, BE and CF (to a given Radius) may be of a given Magnitude.

Let AH be perpendicular to the chord BC; also let BH (= CH) = a , AH = b , BE $\times$ CF = c^2 , and HD = x ; supposing AG to intersect BC in D.



Because of the similar triangles DAH, DBE, and DCF,

we have $\begin{cases} AD(\sqrt{bb+xx}) : AH(b) :: BD(a+x) : BE \\ AD(\sqrt{bb+xx}) : AH(b) :: CD(a-x) : CF. \end{cases}$

Whence, by compounding the two proportions,
 $bb+xx : bb :: a+x \times a-x : BE \times CF = \frac{bb \times aa - xx}{bb+xx} = c^2.$

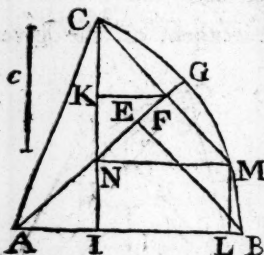
From which equation x is found $= b \sqrt{\frac{aa - cc}{bb + cc}}.$
 By means of which and the foregoing proportions both BE and CF will become known.

Geometrically.

If CI be supposed perpendicular to AB , and FK to CI ; then, the opposite angles CNF and ANI being equal, their complements NCF and NAI will likewise be equal: and therefore, the triangles CFK and ABE being equiangular, we have $AB : BE :: CF : FK$, or $AB \times FK = BE \times CF = c^2$ (according to observation 5.) whence FK is given. Therefore if, from the extremity of the chord CFM , there be drawn MN perpendicular to CI , the length thereof, being twice that of FK , will also be given; and, from thence the following construction.

Take IL a third proportional to $\frac{1}{2}AB$ and the side (c) of the given square, expressing the magnitude of the proposed rectangle: draw LM perpendicular to AB , meeting the arch BC in M ; then a line AG drawn to bisect MC will divide the angle BAC as required.

The numerical solution, from hence, is very concise and easy: for, having found the value of IL (by dividing the measure of the given rectangle by half the radius) let the co-sine AI of the whole, given, angle be added thereto; then the sum AL will be the co-sine of (BM) the difference of the two, required, parts.— This problem becomes impossible when IL is given greater than IB ; that is, when the rectangle proposed is greater than half the rectangle under AB and BI .

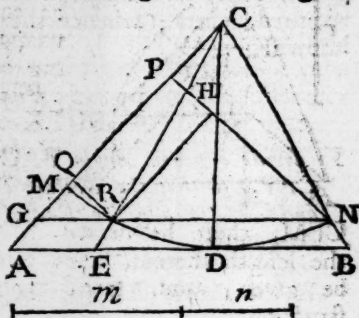


PROBLEM XXXIX.

To divide a given Angle MCN into two such Parts MCD, NCD, that their Tangents AD, BD, may obtain a given Ratio; suppose that of m to n .

Put the radius $CD = a$, the tangent of the given angle $MCN = b$, and that (AD) of the part $MCD = x$. Then it will appear, from Problem 25, that the tangent BD of the remaining part NCD is truly expressed by

$$\frac{a^2 \times b - x}{aa + bx}.$$



Hence we have $m : n :: x : \frac{a^2 \times b - x}{aa + bx}$; and therefore

$nx \times aa + bx = ma^2 \times b - x$. From which x is found

$$= a \sqrt{\frac{m}{n} + \frac{m + n \times a^2}{2nb}} - \frac{n + m \times a^2}{2nb}.$$

Geometrically.

The ratio of AD to BD being given, as m to n , the ratio of their sum AB to their difference AE (supposing $DE = DB$) will be given, as $m + n$ to $m - n$. And, if NG be drawn parallel to BA , meeting CE and CA in R and G , the whole line NG will be to the part GR in the same ratio of $m + n$ to $m - n$; and it is evident, that, if two perpendiculars NP and RQ be let fall from the points N and R upon AC , they will likewise be in that ratio. Whence the following construction.

In NP , perpendicular to MC , take PH a fourth proportional to $m + n$, $m - n$, and PN ; draw HR parallel

Geometrically.

If a line DF be drawn to make the angle BDF equal to CAE, the triangles BDF and CAE will be similar; and we shall, therefore, have $AC \times BF = BD \times CE = c^2$ (according to Observation 5. p. 88) whence BF, and consequently AF, will be given. Moreover the sum of the angles BAD and BDF being equal to the given angle BAC (by hypothesis) and the same sum + ADF equal to a right-angle (by Elem. Cor. 4. to 9. 1.) it is evident that ADF is equal to the difference between the said given angle BAC and a right-one.

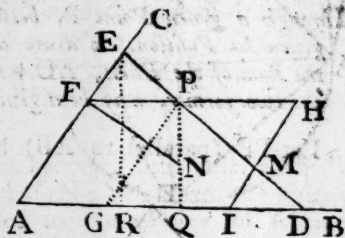
Therefore, having taken BF a third proportional to AB and the side (c) of the given square, make the angle AFO = the angle FAC; and from (O) the intersection of AC and FO, let a circle be described through A and F, intersecting the tangent BK in D, and D; from either of which points draw AD, and the thing is done.

In order to the trigonometrical calculation, let the diameter GH be drawn to bisect the arch AF in H, and let DM be perpendicular thereto: then, having found AF, it will be, as AN ($\frac{1}{2}$ AF) : DM (NB) :: sine AH (co-sine NAO) : sine GD (=co-sine $\frac{1}{2}$ DD) = co-sine of the difference of the two required angles BAD and DAI; whence, as their sum is given, the angles themselves will be known.

This problem becomes impossible, when the circle OAFG neither cuts, nor touches, the line BK; that is, when the given rectangle is greater than the square of the tangent of half the proposed angle, and the angle itself is acute; or, when the said rectangle is less than the squares of half the tangent of the proposed angle, and the angle itself is obtuse.

PROBLEM XLII.

Let PG and PF be parallel to AC and AB, and PQ and ER perpendicular to AB: also let AG and PQ (which are given by the position of P) be denoted by a and b , respectively. Then, calling AD, x , and denoting the given area by c^2 , we



and therefore $\frac{bx}{x-a} \times \frac{x}{2} (=ER \times \frac{1}{2}AD) = c^2$.

Whence $xx - \frac{2c^2x}{b} = -\frac{2ac^2}{b}$; and consequently $x = \frac{c^2 + c\sqrt{c^2 - 2ab}}{b}$.

Geometrically.

If upon AF, a parallelogram AFHI be constituted, to contain the given area, it will appear, by taking away AFPMI from each of the equal quantities AH and ADE, that the remainders PHM and PFE + IDM will likewise be equal: and so, these three triangles

U
being

being all similar to one another, it follows that PH^2 will be $=PF^2 + ID^2$ (*Elem.* 24. 4.) Whence this construction.

From the center F, with the interval PH, let an arch be described, cutting PQ perpendicular to AB, in N; make $ID = PN$, and draw DPE, and the thing is done. For it is evident that PH^2 (FN^2) is $=PF^2 + DI^2$ (PN^2).—Hence it also appears that the problem will be impossible when PH is less than PF; or when the triangle proposed to be constructed is less than twice the parallelogram AFPG.

PROBLEM XLIII.

Through a given Point P, between two Lines AB, AC given by Position, to draw another Line DE, so that the Sum of the Parts, $AD + AE$, cut off thereby, from the two former, may be a given Quantity.

Let PF (parallel to AB) be denoted by a , and PG (parallel to AC) by b ; also let $AD + AE = c$, and $AD = x$: then, by reason of the parallel lines, we shall have $DG (x - a) : PG (b) :: AD (x) : AE = \frac{bx}{x - a}$.

And therefore $\frac{bx}{x - a} + x (=AE + AD) = c$.

From which, by reduction, $xx - c + a - b \times x = -ac$;

and consequently $x = \frac{c + a - b}{2} + \sqrt{\frac{(c + a - b)^2}{4} - ac}$.

Geometrically.

Geometrically.

The triangles DGP and PFE being similar, the rectangle under DG and FE will therefore be equal to the given rectangle under AF and $AG = \overline{AR}^2$; by taking $AM = AF$, and making AR a mean proportional between AM and AG. Moreover, if MN be taken = the given sum of AD and AE, it is manifest that the sum of DG and FE (whose rectangle is above given) will, also, be given = GN.

Therefore, having described a semi-circle upon GN, let RS be drawn parallel to AB, intersecting the periphery thereof in S; from which point, upon AB, let fall the perpendicular SD, and through P draw DPE, and the thing is done. For $DG \times DN = \overline{DS}^2 = \overline{AR}^2 = AM \times AG = AF \times AG = DG \times EF$: whence $EF = DN$; and consequently $AD + AE (= AD + AF + DN) = MN$.

It is plain, this problem becomes impossible, when RS falls below the semi-circle GN; or, when the sum of AD and AE is supposed to exceed that of AG and AF by less than the double of a mean proportional between AG and AF.

Much after the same manner the problem will be resolved, when the difference, the ratio, or the rectangle of AD and AE is given.

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PROBLEM XLIV.

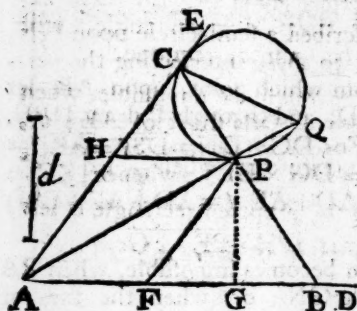
Through a given Point, *P*, between two Right-lines *AD*, *AE*, given by Position, so to draw a Right-line *BPC* that the Rectangle (*BP* × *CP*) under the Parts thereof, intercepted by that Point and those Lines, may be of a given Magnitude.

Let *PF* and *PH* be parallel to *AC* and *AB*, and *PG* perpendicular to *AB*:

then, calling *PF*, *a*; *PH*, *b*; *FG*, *c*; and *BF*, *x*; we have $\overline{BP}^2 = aa + xx - 2cx$ (Elem. 10. 2.)

And, by similar triangles, *BF* (*x*) : *BP* ::

PH (*b*) : *CP* = $\frac{b}{x} \times BP$.



Consequently $BP \times CP = \frac{b}{x} \times BP^2 = \frac{b}{x} \times aa + xx - 2cx$. Hence, if the given value of $BP \times CP$ be denoted by d^2 , then will $\frac{d^2 x}{b} = aa + xx - 2cx$: which equation, solved, gives

$$x = \frac{dd}{2b} + c \pm \sqrt{\left(\frac{dd}{2b} + c\right)^2 - a^2}.$$

Geometrically.

The rectangle under two unknown lines being given, another line must therefore be found, or assumed, under which and some given line in the figure, an equal rectangle may be contained. (Vid. Obser. 5. p. 88.)

As,

As, in the present case, the line AP is given, both in length and position, let the rectangle under it, and a part, PQ , of the same line produced, be therefore assumed $= RP \times CP$; then the consequence will be, that, besides obtaining $PQ = \frac{d^2}{AP}$, the triangles PQC and PBA (supposing QC drawn) will also be similar because $AP : BP :: CP : PQ$. And so, the angle PCQ being $=$ the given angle PAB , it is evident that a segment of a circle described upon PQ (by *Elem. 22. 5.*) capable of containing the said given angle, will intersect AE in the point (or points) required.

This problem will, it is manifest, be impossible, when the circle, described as above, falls short of AE ; or, according to the algebraic solution, when $\left[\frac{ad}{2b} + c\right]^2 - a^2$ is negative; that is, when the proposed rectangle is less than $2b \times a - c$, or, its equal $2PH \times PF - FG$.

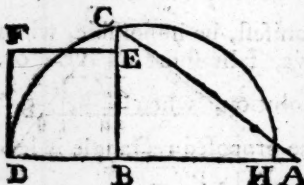
PROBLEM

PROBLEM XLV.

The Area (a^2) of a right-angled Triangle ABC, whose Sides are in Arithmetical Progression, being given, to determine the Triangle.

Put the greater leg $AB=x$, and the common difference $=y$; so shall $BC=x-y$, and $AC=x+y$; and therefore

$$\left\{ \begin{array}{l} (x+y)^2 = (x-y)^2 + x^2 \\ \frac{1}{2}x \times (x-y) = a^2 \end{array} \right\} \text{ by the question.}$$



From the former of which equations we have $x + 2xy + yy = 2xx - 2xy + yy$; and consequently $4y = x$: whence, by substituting for x in the second equation, we get $2y \times 3y = a^2$: from

which y is given $= \sqrt{\frac{a^2}{6}}$; and $x (=4y) = 4\sqrt{\frac{a^2}{6}}$.

Therefore $BC = 3\sqrt{\frac{a^2}{6}}$, $AB = 4\sqrt{\frac{a^2}{6}}$, and $AC = 5\sqrt{\frac{a^2}{6}}$.

Geometrically.

It is well known that $\overline{AC+BC} \times \overline{AC-BC} (= \overline{AC}^2 - \overline{BC}^2)$ is $= \overline{AB}^2$ (*Elem. Corol. to 8. 2.*) And, by the question, $AC + BC$ is $= 2AB$ (because AB is an arithmetical mean between AC and BC .) Therefore $2AB \times \overline{AC-BC} = \overline{AB}^2$; and consequently $AC - BC = \frac{1}{2}AB$: take these equal quantities from the equal quantities $AC + BC$ and $2AB$, and the remainders, $2BC$ and $\frac{1}{2}AB$, will be equal; and consequently $3AB = 4BC$. But $\frac{1}{2}AB \times \overline{BC} (= \frac{1}{2}BC \times \overline{BC}) = \overline{BD}^2 (= a^2)$; and therefore $\overline{BC}^2 = \frac{1}{3}\overline{AB}^2$.

Hence.

Hence, if BH be taken to BD (*a*) in the proportion of 3 to 2, a mean proportional BC between DB and HB, will be the lesser leg of the triangle required; whose greater leg BA, being in proportion thereto, as 4 to 3, is also given from hence.

PROBLEM XLVI.

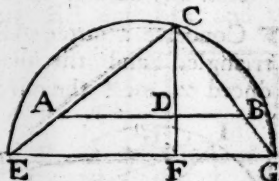
The Area (a^2) of a right-angled Triangle ABC, whose Sides are in Geometrical Proportion, being given; to determine the Triangle.

Make $AC = x$; then $BC = \frac{2aa}{x}$, and $AB =$

$$\sqrt{x^2 + \frac{4a^4}{xx}}$$

Therefore, $\frac{2a^2}{x} : x :: x :$

$$\sqrt{x^2 + \frac{4a^4}{xx}}, \text{ by the quest.}$$



$$\text{Hence } x^2 = \frac{2a^2}{x} \sqrt{x^2 + \frac{4a^4}{xx}}; x^4 = \frac{4a^4}{xx} \times \frac{x^2 + 4a^4}{xx};$$

$$x^8 - 4a^4x^4 = 16a^8; x^4 - 2a^4 = a^4\sqrt{20}; \text{ and } x = a \times$$

$$\sqrt[4]{2 + \sqrt{20}}.$$

Geometrically.

Since, by hypothesis, $AB : AC :: AC : BC$, therefore is $\overline{AB}^2 : \overline{AC}^2 :: \overline{AC}^2 : \overline{BC}^2$ (*Elem. Cor. 1. to 11. 4.*)

But $\overline{AC}^2$, supposing CD perpendicular to AB, is equal to $AB \times AD$; and $\overline{BC}^2$ equal to $AB \times BD$ (*Elem. Cor. to 19. 4.*)

Therefore $\overline{AB}^2 : AB \times AD :: AB \times AD : AB \times BD$; or, $AB : AD :: AD : BD$.

Whence the following construction.

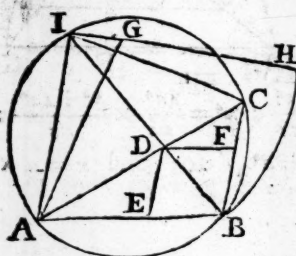
Draw any line EG, at pleasure; which divide at F (by *Elem. 19. 5.*) according to extreme and mean proportion (so that $EG : EF :: EF : GF$); erect the perpendicular

perpendicular FC, and upon EG let a semi-circle be described intersecting it in C; join E, C, and G, C; and let AB (by Prob. 41.) be drawn, parallel to EG, to cut off the given area ABC ($=a^2$), and the thing is done. For it is manifest that AB is divided, by CDF, in the same proportion with EG; or that, $AB : AD :: AD : BD$, as above.

PROBLEM XLVII.

Supposing one Side AC, and the opposite Angle ABC of a Triangle to be given, together with the Side DE, or DF, of the inscribed Rhombus EF; to find from thence the other two Sides of the Triangle.

Conceive a circle ABCI to be described about the triangle, and the diagonal of the rhombus produced to meet the circumference thereof in I; also



let AI and CI be drawn; which will be equal to each other (*Elem.* 12. 3.) as being the subtenses of the equal angles IBA and IBC, formed by the diagonal and sides of the rhombus: and since the arcs AI and CI, as well as the chords, are equal, the angles IAD and

IBA, insisting upon them, must likewise be equal; and consequently the triangle IAD similar to the triangle IBA.

But the vertical angle AIC of the isosceles triangle ACI (as well as the base AC) is given, being $=2$ right-angles $-ABC$ (*Elem. Cor.* 17. 3.) whence IA will be given (*by plane trigonometry.*)

Put, therefore, $IA=a$, $BD=b$, and $IB=x$; then, from the similarity of the triangle above specified, we shall have, $x-b$ (ID) : a (IA) :: a (IA) : x (IB) whence $xx-bx=aa$; and consequently $x=\sqrt{aa+\frac{1}{4}bb}+\frac{1}{2}b$. From which, and the known values of IA and the angle ABI, the value of AB, &c. will also be known.

Geometrically.

Geometrically.

The geometrical construction hereof, as the rect angle $BI \times DI$ (by the foregoing proportion) is given $\sqrt{AI^2}$, is obvious from Problem 6; and is thus,

Having, upon the given side AC , described a segment of a circle ABC capable of containing the given angle ABC (*Elem.* 22. 5); and, in the other segment, AIC , constituted the isosceles triangle AIC , make IG perpendicular to IA , and equal to half (BD) the given diagonal of the rhombus; also, in IG produced, take GH equal to the distance GA ; and, through H , from the center I , let an arch be described intersecting the circle ABC in B ; then draw BA and BC , and the thing is done.

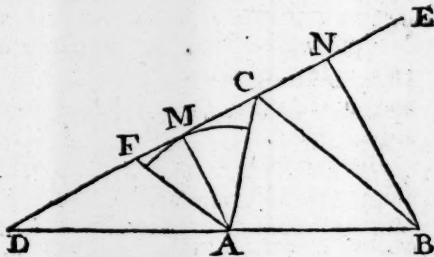
PROBLEM XLVIII.

From two given Points A and B , to draw two Lines AC and BC , to meet in a Right-line DE given by Position, and form an Angle ACB of a given Magnitude.

Make AM and BN perpendicular to DE ; and let AF be supposed

parallel to BC : then, calling AM , a ; BN , b ; MN , c ; and MC , x ; it will be as $b : c - x$ $(NC) :: a : MF$

$$= \frac{ac - ax}{b}.$$



But MF and MC are tangents of the angles MAF and MAC ; which angles together (because of the parallel lines AF and BC) make an angle CAF equal to the given angle ACB : therefore, if the tangent of the said given angle, to the radius AM , be denoted

X

noted

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noted by t , it will appear, from Problem 25, that

$$\frac{MC + MF \times AM^2}{AM^3 - MC \times MF} = t.$$

This, in species, gives
$$\frac{x + \frac{ac - ax}{b} \times a^2}{a^2 - \frac{acx - axx}{b}} = \frac{bx - ax + ac \times a}{ab - cx + xx} = t.$$

Which, by reduction, becomes
$$\frac{b - a \times ax}{t} + \frac{aac}{t} = xx - cx + ab.$$
 Whence, making $c + \frac{b - a \times a}{t} = d$,

and $a \times \frac{ac}{t} - b = f$, we have $f = xx - dx$; and consequently
$$x = \frac{1}{2}d \pm \sqrt{f + \frac{1}{4}dd}.$$

When the line MN is parallel to that joining the given points A, B; then, BN (b) becoming = AM (a) we have, in this case, $d = c$, and $f = \frac{aac}{t} - a^2$; and

therefore
$$x = \frac{c}{2} + \sqrt{\frac{aac}{t} - ac + \frac{1}{4}cc}.$$
 Which may serve as a theorem for finding the segments of the base of a triangle (and consequently the triangle itself) when the whole base, the perpendicular, and the vertical angle are given.

As to the geometrical construction of the general problem, it is extremely obvious; since a segment of a circle described upon AB (*by Elem. 22. 5.*) capable of containing the given angle, will intersect DE in the point (or points) required. Whence it also appears that the problem will be impossible when the circle falls short of the line FE; and, consequently, that the angle ACB will be the greatest possible when the circle touches the said line; or, when DC is a mean proportional between DA and DB (*Elem. Cor. to 22. 3.*)

PROBLEM

PROBLEM XLIX.

To find a Point C, in a Right-line DE, given by Position; so that two Lines CA, CB being drawn from thence to two given Points A and B, the one BC shall exceed the other AC by a given difference d.

Let AR and BS be perpendicular to DE; and let these perpendiculars, together with RS (which are all known from the position of DE) be denoted by a , b , and c , respectively: then putting $AC = x$, we have $RC (= \sqrt{AC^2 - AR^2})$

$$= \sqrt{xx - aa}; \text{ and } SC (= \sqrt{BC^2 - BS^2}) = \sqrt{xx + 2xd + dd - bb} = c - \sqrt{xx - aa}.$$

From which equation, by squaring both sides thereof, we get $xx + 2dx + dd - bb = cc - 2c\sqrt{xx - aa} + xx - aa$.

This contracted becomes $2c\sqrt{xx - aa} = bb - aa - dd + cc - 2dx$, or, $\sqrt{xx - aa} = m - nx$; by dividing by $2c$, and putting $\frac{bb - aa - dd + cc}{2c} = m$, and $\frac{d}{c} = n$.

Therefore, by squaring again, $xx - aa = mm - 2mnx + nnxx$; whence $1 - nn \times xx + 2mnx = aa + mm$, or $xx + \frac{2mnx}{1 - nn} = \frac{aa + mm}{1 - nn}$.

And consequently $x = \sqrt{\frac{aa + mm}{1 - nn} + \frac{m^2 n^2}{(1 - nn)^2}} - \frac{mn}{1 - nn}$.

But, if KL be supposed parallel to AC, meeting HA produced (if need be) in L, it is manifest that $HC : AC :: HK : KL$. Therefore $KL = AB$, whence the following construction.

Having taken $FG = \frac{PQ^2}{2AB}$, $GI = PQ$, and erected the perpendiculars GH and IK, and also drawn HA (as above specified), from the center K, with the interval AB, let an arch be described, cutting HA produced in L; and, having drawn LK, make AC parallel thereto; which will cut the given line DE in the point required.

The trigonometrical calculation, from this construction, may be as follows.

Having computed $FG (= \frac{PQ^2}{2AB})$ and subtracted it from FD and FA, the remainders GD and GA will be given; and it will then be as $GD : GA :: \text{tang. DHG (or co-tang. D)} : \text{tang. AHG}$; whence the angle LHK is likewise known. And, since $\sin. DHG$ (or co-sin. D) : $\text{rad.} :: IG (PQ) : HK$, the value of HK (as well as those of KL and the angle KHL) will be known: from which the other two angles of the triangle HKL being found, the angle DAC ($= ECA - D = HKL - D$) will also be obtained.

After the very same manner the problem may be resolved, when the sum, instead of the difference, of the lines AC and BC is given.

As to the restrictions of the last problem, it is evident that the given difference must never exceed the distance AB: but when the line DE passeth between the points A and B, the limit will be still less; but is easily determined, in any case, from the given position of DE. It is a little remarkable, that the above solution fails in that particular case, only, wherein the general problem becomes most simple; that is, when DE is perpendicular to AB. But here the operation will, also, become more simple and expeditious: for the

$$\frac{AC \times \overline{BD}^2}{AB} = AC \times BC + AC \times BG \quad (\text{by taking } BG = \frac{\overline{BD}^2}{AB}) \\ = AC \times \overline{BC} + BG = AC \times CG.$$

Therefore, having taken BG a third proportional to AB and BD; and CL a mean-proportional between AC and CG; draw BK perpendicular to GF, meeting the circumference of a semi-circle, described upon AC, in K; and, having drawn AKM, and taken AH equal to the given difference of AD and CD, upon H as a center, with the radius LC, let an arch be described, intersecting AK in M; from which point upon GF let fall a perpendicular MN; then, if from A, with the radius AN, another arch be described, it will intersect the arch of the given semi-circle in the point, D, required.

For $AB : BC :: \overline{AB}^2 : AB \times BC (= \overline{BK}^2)$, *Elem.*
Cor. 19. 4. $:: \overline{AN}^2 = (\overline{AD})^2 : \overline{NM}^2 = \overline{AD}^2 \times \frac{BC}{AB}$.
 And $\overline{NM}^2 = (\overline{AD})^2 \times \frac{BC}{AB} + \overline{HN}^2 = \overline{HM}^2 = AC \times CG$
 (by constr.) $= \overline{AD}^2 \times \frac{BC}{AB} + \overline{CD}^2$ (p. above.)

Therefore $CD = HN = AN + AH = AD + AH$; and consequently $CD - AD = AH =$ the given difference, by construction.

The method of calculation, from this construction, is sufficiently easy: for having computed $BG (= \frac{\overline{BD}^2}{AB})$

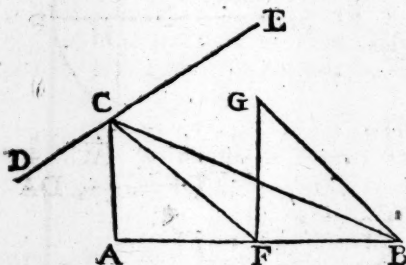
and $HM (= \sqrt{CA \times \overline{CB} + \overline{BG}})$ and also the angle BAK (which is had by the proportion $AB : BC :: \text{sq. rad.} : \text{sq. tang. BAK}$); you will then have, in the triangle HAM, two sides and one angle; whence every thing else is readily determined.

PROBLEM

PROBLEM LI.

From the Middle F, and the two Extremes A and B, of a given Right-line AB, to draw three Lines to meet in a Point C, in a Right-line DE given by Position, so as to be in Geometrical Proportion, or so that $AC : FC :: FC : BC$.

Because $\overline{BC}^2 + \overline{AC}^2 = 2\overline{FC}^2 + 2\overline{BF}^2$ (Elem. 11. 2.) And $BC \times AC = \overline{FC}^2$ (by Hyp. and Elem. 10. 4.) it is evident that $\overline{BC}^2 - 2BC \times AC + \overline{AC}^2 = 2\overline{BF}^2$, or $\overline{BC - AC}^2 = 2\overline{BF}^2 = \overline{BG}^2$, by taking BG as the diagonal of the square whose side is BF.



Hence $BC - AC$ is given $= BG$: and so the case in question is reduced to Problem 49; to which I shall therefore refer for the remaining part of the solution.—It ap-

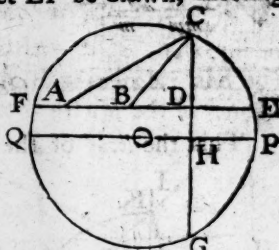
pears from hence that a point in the circumference of a given circle, whose center is in the line AB, may be so determined, by the last problem, that three lines drawn from thence to the three given points A, F, B, shall be in geometrical proportion.

PROBLEM

PROBLEM LII.

From two given Points A, B, within a given Circle, to draw two Lines AC, BC, to meet in the Periphery thereof, so that the Sum of their Squares may be a given Quantity.

Through the given points let EF be drawn, meeting the circumference of the circle in E and F; parallel to which, draw the diameter PQ; and let the chord CG be drawn to cut EF and PQ, at right-angles, in D and H.



Put $EF = a$, $EA = b$,
 $EB = c$, $DH = d$, $ED = x$,
 and $DC = y$; and let the given quantity, $AC^2 + BC^2$, be denoted by e^2 : then will $DF = a - x$, $DA = b - x$, $DB = c - x$, $DG = y + 2d$.
 But $ED \times DF = CD \times DG$ (Elem. 21. 3.)
 And $DA^2 + DC^2 + DB^2 + DC^2 = AC^2 + BC^2$ (Elem. 8. 2.) Which, in species,

$$\text{give } \begin{cases} x \times a - x = y \times y + 2d \\ [b - x]^2 + y^2 + [c - x]^2 + y^2 = e^2. \end{cases}$$

$$\text{or } \begin{cases} ax - xx = yy + 2dy \\ 2xx + 2yy - 2bx - 2cx = ee - bb - cc. \end{cases}$$

Whence, by adding the double of the former equation to the latter, we get $2ax - 2bx - 2cx = ee - bb - cc + 4dy$;

$$\text{and consequently } y = \frac{bb + cc - ee}{4d} + \frac{a - b - c}{2d} \times x = f + gx,$$

$$\text{by making } \frac{bb + cc - ee}{4d} = f, \text{ and } \frac{a - b - c}{2d} = g.$$

Y

Now,

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Now, the value of y thus found being substituted in the first equation, there arises $ax - xx = ff + 2fgx + g^2x^2 + 2df + 2dgx$;

or, $1 + gg \times xx + 2dg + 2fg - a \times x = -ff - 2df$.

From which, by making $m = 1 + gg$, $n = a - 2dg - 2fg$,

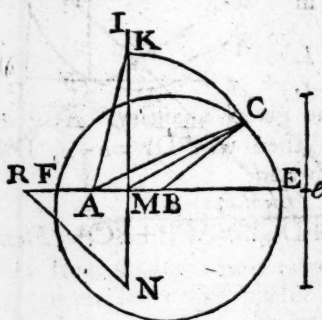
and $p = f \times f + 2d$, the value of x is found $= \frac{n}{2m} \pm$

$$\sqrt{\frac{nn}{4mm} - \frac{p}{m}}.$$

Geometrically.

That the *locus* of the vertex of a triangle, whereof

the base and sum of the squares of its two sides are given, is the circumference of a circle, described from the middle of the base as a center, it is evident, from *Elem. 11. 2*; because the line drawn from the vertex to the middle of the base, is an invariable quantity.



Therefore, having bisected AB with the perpendicular IMN , take MN and MR equal, each, to half the side (e) of the given square; draw NR , and, from A to the perpendicular MI , draw AK equal to NR ; and upon the center M , at the distance MK , let an arch be described; which will meet the circumference of the given circle in the point C , required. For AC , BC , and MC being drawn, it is evident that $\overline{AC}^2 + \overline{BC}^2 = 2\overline{AM}^2 + 2\overline{CM}^2 = 2\overline{AM}^2 + 2\overline{MK}^2 = 2\overline{AK}^2 = 2\overline{RN}^2 = 4\overline{RM}^2 = e^2$. In this problem it is requisite

that $MC (= \sqrt{\frac{1}{2}e^2 - \overline{AM}^2})$ should be greater than the difference, and less than the sum, of the radius of

$\frac{dag}{bc+gg}$, the value of x is found $= \frac{1}{2}d + \sqrt{\frac{1}{4}dd - fg}$.

Whence every thing else is readily determined.

Geometrically.

If QM be supposed parallel to AB, the triangle CQM will, it is plain, be similar to ABF; and consequently $AB \times QM = BP \times CQ$ (*Elem.* 24. 3.) $= g^2$: whence QM is given; and, from thence the following construction.

Find a third-proportional to AB and the side (g) of the given square; and, in BF produced, take FD equal to the double thereof; draw DG parallel to FC, intersecting the circumference of a circle, described from the center A with the radius AC, in the point G; and, having drawn CG, draw PQ perpendicular thereto.

For the trigonometrical calculation, it will be $AF : AD (AF + FD) :: \text{co-f. HGC} : \text{co-f. HG}$, the difference between the angles BAP (HAQ) and CAQ: from which, as their sum is given, the angles themselves will be known.—This problem becomes impossible when FD is greater than FH; that is, when the proposed rectangle is greater than half the rectangle under AB and FH.

PROBLEM

PROBLEM LIV.

Two Lines AB, AC, drawn from the same Point A, being given both in Position and Length; to draw another Line PQ through that Point, so that two Perpendiculars BP, CQ, falling thereon from the Extremes of the two given Lines, may form two Triangles ABP, ACQ, equal to each other.

Let BA and CQ be produced to meet in E; and upon BE let fall the perpendicular CF: and put $CF = a$, $AF = b$, $AB = c$, and $EF = x$.

The areas of similar triangles being, in proportion, as the squares of their homologous sides; and both the triangles AEQ, ABP, being similar to ECF, we have $CE^2 (aa + xx) :$

$$\frac{1}{2}CF \times EF (\frac{1}{2}ax) :: AB^2 (c^2) : \text{area ABP} = \frac{\frac{1}{2}a^2x}{aa+xx}.$$

And, $aa + xx : \frac{1}{2}ax :: x+b)^2 (=AB^2) : \text{area AEQ} = \frac{\frac{1}{2}ax \times x+b)^2}{aa+xx}$. This, taken from the area AEC (=

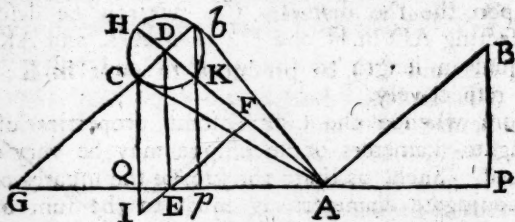
$$\frac{1}{2}a \times x+b) \text{ leaves the area ACQ} = \frac{\frac{1}{2}a \times x+b \times aa-bx}{aa+xx}.$$

Which being equal to ABP, by hypothesis, we therefore have $x+b \times aa-bx = c^2x$: from whence, making $d = \frac{cc-aa}{b}$, x is found $= \sqrt{aa + \frac{1}{2}dd} - \frac{1}{2}d$.

Geometrically.

Geometrically.

If Ab be supposed perpendicular, and equal, to AB , and bp perpendicular to PQ ; the triangle Abp , being similar to ABP , will also be equal to it; and consequently equal to ACQ : and, if these equal triangles be, successively, taken from the trapezium $AQCb$, the remainders $pbcQ$, and ACb will likewise be equal.



Hence it appears that the parallelogram under Cb and DE , whose angle is CDE (supposing DE parallel to, and an arithmetical mean between, CQ and bp) is equal to the given triangle ACb ; and consequently the altitude thereof equal to half that of the triangle. Whence it is evident that the point E must fall, somewhere, in a line FI drawn through the middle of AD (or AC) parallel to bC (*Elem. 2. 2.*) But the point E , since the angle AED is a right-one, will likewise fall in the circumference of a semi-circle described upon the diameter AD (*Elem. 13. 3.*) And therefore FE , being a radius, must be equal to AF ; and consequently $DG = AD$; supposing DC produced to meet AQ in G .

Therefore, in order to the geometrical construction, having made Ab perpendicular, and equal to, AB , and drawn AD to the middle of Cb (as above intimated) let DG , in DC produced, be taken equal to AD ; and from G , through A , draw GP , and the thing is done.

It often happens that the demonstration of a geometrical construction, to be the most neat and elegant, proceeds upon principles very different from those whereby we first arrived at such construction. The case

case above is an instance of it: where, from the similar triangles, it is manifest that $GQ (Ap) : QC :: Gp (AQ) : pb$; and therefore $\frac{1}{2}AQ \times QC = \frac{1}{2}Ap \times pb = \frac{1}{2}BP \times AP$. Q. E. D.

Note. If AB and AC be two semi-conjugate diameters of an ellipsis, then the line PAQ, determined as above, will be the position of the greater axis: and, if, upon the the diameter Cb, a circle be described intersecting AD in H and K; then AH, and AK will be equal in length to the greater, and lesser, semi-axis, respectively.

From whence the most useful properties of the conjugate diameters of an ellipsis may be very easily deduced. Such, as that, the sum of the squares of any two conjugate diameters, is equal to the sum of the squares of the two axes: and that, any parallelogram described about the conjugate diameters of an ellipsis is equal to the rectangle under the two axes; and so forth. But these are matters not altogether proper to be insisted on in this place.

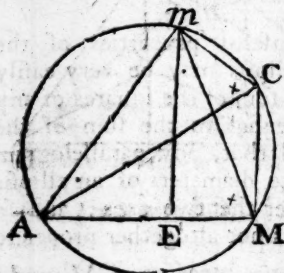
However it will not be improper to observe, that the last problem is always possible, except in those two cases, wherein the given lines are perpendicular, and form one continued line.

PROBLEM

PROBLEM LV.

If, at the Extremes, M and m , of two given Right-lines AM and Am , making a given Angle MAM , two Perpendiculars, MC and mC be erected: it is proposed to find the Distance of their Intersection, C , from A the given angular Point.

Since the angles AMC and AmC are both right ones, the circumference of a circle, described upon the



diameter AC , will pass through M and m . Therefore, if Mm be drawn, and a perpendicular mE be let fall upon AM , the triangles AmC and mEM will appear to be equiangular; because the angles ACm and EMm , inscribing on the same arch Am , are equal; and AmC

is equal to mEM , being both right angles.

If now the ratio of mE to Am (which is given because the angle EAm is given) be denoted by that of s to r ; and the ratio of AE to Am by that of c to r ;

we shall have $AE = \frac{c}{r} \times Am$; and therefore $\overline{Mm}^2$ (= *Elem. II. 17*)

$$\overline{AM}^2 + \overline{Am}^2 - 2AM \times AE = \overline{AM}^2 + \overline{Am}^2 - \frac{2c}{r}$$

$\times AM \times Am$: whence, by reason of the similar triangles above specified, it will be, $\overline{mE}^2 : \overline{mA}^2$ ($:: s^2 : r^2$)

$$:: \overline{Mm}^2 : \overline{AC}^2 = \frac{r^2}{s^2} \times \overline{AM}^2 + \overline{Am}^2 - \frac{2rc}{ss} \times AM$$

$\times Am$; whence AC is given.

As to the geometrical construction, it is indicated by the conditions of the problem, without any sort of argumentation.

PROBLEM

PROBLEM LVI.

To find a Point C, from whence three Right-lines drawn to so many given Points A, B, and E, shall obtain the Ratio of three given Quantities a, b, and c, respectively.

The given points being joined; make $AB=f$, $AE=g$, and $AC=x$: then,

BC being $=\frac{bx}{a}$, and EC

$=\frac{cx}{a}$ (by hypothesis) we

also have $AM (= \frac{1}{2}AB$

$+ \frac{AC^2 - BC^2}{2AB}) = \frac{f}{2} +$

$\frac{aa - bb \times x^2}{2aaf}$ (Elem. 9. 2. Cor. 2.); and $Am (= \frac{1}{2}AE +$

$\frac{AC^2 - EC^2}{2AE}) = \frac{g}{2} + \frac{aa - ee \times x^2}{2aag}$; supposing CM and

Cm to be perpendicular to AB and AE , respectively.

Hence, putting $\frac{aaf}{aa - bb} = b$, $\frac{aag}{aa - ee} = k$, and $z =$

$\frac{f}{2} + \frac{aa - bb \times x^2}{2aaf}$ ($= AM$) $= \frac{f}{2} + \frac{xx}{2b}$, we get $xx =$

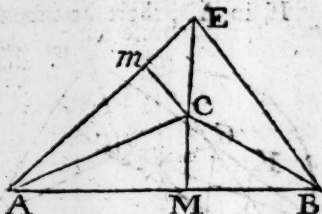
$2bx - fb$; and therefore $Am (= \frac{g}{2} + \frac{xx}{2k}) = \frac{g}{2} +$

$\frac{2bx - fb}{2k} = l + \frac{bx}{k}$; by making $l = \frac{g}{2} - \frac{fb}{2k}$.

Now, since (by the last problem) $\overline{AM}^2 + \overline{Am}^2 = \frac{zc}{r} \times AM \times Am = \frac{ss}{rr} \times \overline{AC}^2$ (where s and c denote the sine and co-sine of the given angle MAm , to the radius r) we shall, from hence, by substituting the

Z

above



above values of AM , A_m , and AC , obtain the following equation,

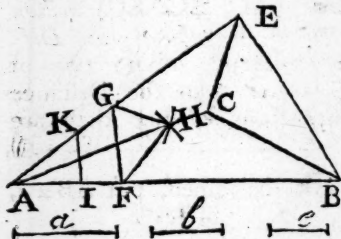
$$x^2 + l + \frac{bx}{k} - \frac{2cx}{r} \times l + \frac{bx}{k} = \frac{ss}{rr} \times 2bx - fh.$$

From the resolution of which the value of z , and from thence the position of the point D, will be determined.

Geometrically.

If, in AB , there be taken $AF=a$, and FH be drawn,

to make the angle $\angle AFH = \angle ACB$ and meet AC in H ; then, the ratio of FH to AF (by reason of the similar triangles $\triangle AFH, \triangle ACB$) being the same with that of BC to AC , it is evident that FH is given $= b$.



Moreover, if HG be drawn, making the angle $\text{AHG} = \text{AEC}$, it will likewise appear, that both HG and AG are given.

For, since $AG : AH :: AC : AE$,

And $AF : AH :: AC : AB$;

it follows that $AG : AF :: AB : AE$; whence AG is given: and then it will be $a : e (:: AC : EC) :: AG : HG$. Whence HG is given; and, from thence, the following

Construction.

Take $AF=a$, and make the angle $AFG=AEB$; also take $AI=\epsilon$, and draw IK parallel to FG ; moreover, from the centers F and G , with the intervals b and AK , describe two arcs, and from the point H of their intersection draw HF and HA , then a line, BC , drawn to make the angle $ABC=AHF$, will cut AH , produced, in the point required.

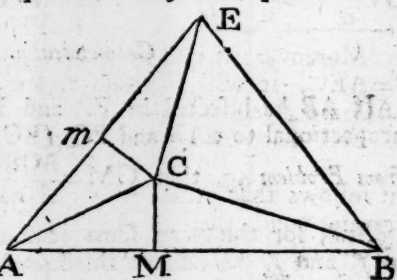
As to the trigonometrical solution, it is too obvious, from the construction, to need an explanation. But it

it will be proper to take notice that the problem itself becomes impossible, when the two arcs, described from the centers F and G, neither cut, nor touch, each other; that is, when the distance FG is, either, greater than the sum, or less than the difference, of b and AK.

PROBLEM LVII.

Three Points, A, B, and E, being given, to determine a fourth Point C, so that Lines (AC, BC, EC) drawn from thence to the three former, may have given Differences. (Provided the Difference of no two of the said Lines be given greater than the Distance of the two given points from whence they are drawn.)

Supposing the given points to be joined, put $AB = a$, $AE = b$, $AC = x$, $BC = x + p$, and $EC = x + q$ (where p and q represent the given differences.) Then, if upon AB and AE, the perpendiculars CM and Cn be let fall,



it will be (by a known property of triangles) $AB (a) : BC + AC (2x + p) :: BC - AC (p) : BM - AM = \frac{2px + pp}{a}$:

whence $AM = \frac{1}{2}a - \frac{2px + pp}{2a}$: and, by the very

same argument, $Am = \frac{1}{2}b - \frac{2qx + qq}{2b}$. Which two

values, by putting $\frac{1}{2}a - \frac{pp}{2a} = f$, and $\frac{1}{2}b - \frac{qq}{2b} = g$

(for the sake of brevity) will become $f - \frac{px}{a}$, and $g - \frac{qx}{b}$.

Z 2

Moreover,

Moreover, if the sine, and the co-sine of the given angle MAm , to the radius r , be denoted by s and c , respectively, it will appear, from the problem preceding the last, that $\overline{AM}^2 + \overline{Am}^2 - \frac{2c}{r} \times AM \times Am = \frac{ss}{rr} \times$

$$\overline{AC}^2; \text{ or, (in species) } f - \frac{px}{a}^2 + g - \frac{qx}{b}^2 - \frac{2c}{r} \times f - \frac{px}{a} \times g - \frac{qx}{b} = \frac{s^2 x^2}{r^2}. \text{ Which equation may}$$

$$\text{be reduced to } \frac{pp}{aa} + \frac{qq}{bb} - \frac{2cpq}{rab} - \frac{ss}{rr} \times x^2 - \frac{fp}{a} + \frac{gq}{b} - \frac{cgp}{ra} - \frac{cfq}{rb} \times 2x = -f^2 - g^2 - \frac{2cfg}{r}.$$

Whence, by substituting for the coefficients of the powers of x , &c. the value of x , will be found, and from thence the position of the point C .

Geometrically.

If AB be bisected in F , and FG be taken a third proportional to $2AB$ and PQ ($BC - AC$), it is evident, from Problem 49, that $GM = \frac{AC \times PQ}{AB}$. $PQ = GI + P$

And, for the very same reasons, if AE be bisected in f , and fg be taken a third proportional to $2AE$ and RS ($EC - AC$) we shall also have $gm = \frac{AC \times RS}{AE}$.

Whence it appears that GM is to gm , in the given ratio of $\frac{PQ}{AB}$ to $\frac{RS}{AE}$, or of PQ to $\frac{AB \times RS}{AE}$, or, lastly, of GI to gi ; by taking $GI = PQ$, and $gi = \frac{AB \times RS}{AE}$.

H

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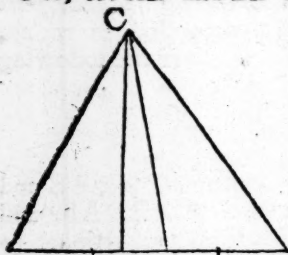
diculars GH, gH; IK, iK, as above specified; let an arch of a circle, from the center K, at the distance of AB be next described, and through A let HL be drawn, meeting it in L; then a right-line AC drawn parallel to that joining the points L and K, will cut ~~it~~ in the point required; as is evident from the above-mentioned problem. By means of which the trigonometrical solution will also be brought out.

In this construction, the solution of the problem for describing a circle to touch three given circles, is determined.

LEMMA.

If from the Vertex of any plane Triangle ABC, a right-line CD be drawn, to divide the Base AB in the Ratio of two given Numbers n and m (so that $AD : BD :: n : m$); then the Sum of the Multiples of the Squares of the two Sides of the Triangle, whose Factors are the said given Numbers, taken alternately, will be equal to the Sum of the Rectangle of the two Parts of the Base and the Square of the dividing Line, repeated as often as there are Units in the two proposed Numbers (that is, m times $\overline{AC}^2 + n$ times $\overline{BC}^2 = m + n$ times $\overline{AD} \times \overline{BD} + \overline{DC}^2$)

For, let AD and BD be bisected in M and N, and



upon AB let fall the perpendicular CE; then will

$$\overline{AC}^2 - \overline{DC}^2 = \overline{AD} \times 2ME$$

(Elem. Cor. 1. to 9. 2.)

Whence it is evident that m times $\overline{AC}^2 - m$ times $\overline{DC}^2 = m$ times $\overline{AD} \times 2ME$.

And, by the very same argument, it appears that

$$n \text{ times } \overline{BC}^2 - n \text{ times } \overline{DC}^2 = n \text{ times } \overline{BD} \times 2NE.$$

Therefore,

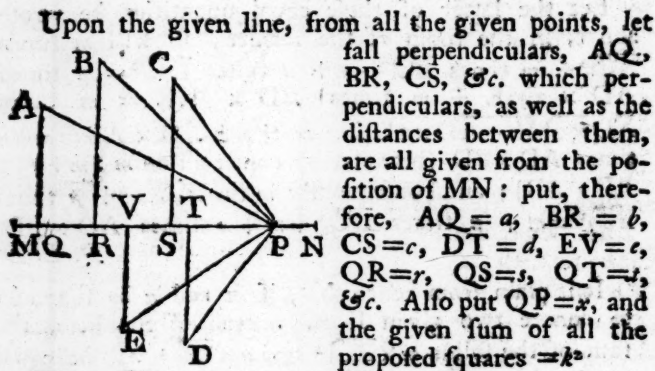
Therefore, by adding these equal quantities together,
 m times $\overline{AC}^2 + n$ times $\overline{BC}^2 - m + n$ times $\overline{DC}^2 =$
 m times $\overline{AD} \times 2\overline{ME} + n$ times $\overline{BD} \times 2\overline{NE}$. But $m :$
 $n :: \overline{BD} : \overline{AD}$ (by hyp.) $\therefore \overline{BD} \times 2\overline{NE} : \overline{AD} \times 2\overline{NE}$
 (Elem. 7. 4.); and therefore n times $\overline{BD} \times 2\overline{NE} = m$
 times $\overline{AD} \times 2\overline{NE}$.

Let the latter of these equal quantities be wrote
 above, in the room of the former; so will m times
 $\overline{AC}^2 + n$ times $\overline{BC}^2 - m + n$ times $\overline{DC}^2 = m$ times
 $\overline{AD} \times 2\overline{ME} + m$ times $\overline{AD} \times 2\overline{NE} = m$ times
 $\overline{AD} \times 2\overline{ME} + \overline{AD} \times 2\overline{NE} = m$ times $\overline{AD} \times \overline{AB} = m + n$
 times $\overline{AD} \times \overline{BD}$ (because, by construction $m : m + n ::$
 $\overline{BD} : \overline{AB}$); and consequently m times $\overline{AC}^2 + n$ times
 $\overline{BC}^2 = m + n$ times $\overline{DC}^2 + m + n$ times $\overline{AD} \times \overline{BD}$.
 Q. E. D.

It is plain from hence that, if m and n be supposed
 to denote two given lines, instead of numbers, the
 sum of the solids $m \times \overline{AC}^2$ and $n \times \overline{BC}^2$ will be equal
 to the sum of the solids $m + n \times \overline{DC}^2$ and $m + n \times$
 $\overline{AD} \times \overline{BD}$.

PROBLEM LVIII.

From any Number of given Points, A, B, C, D, &c. to draw as many Lines AP, BP, CP, &c. to meet in a Right-line MN given by Position, so that the Sum of all their Squares may be a given Quantity.



Upon the given line, from all the given points, let fall perpendiculars, AQ, BR, CS, &c. which perpendiculars, as well as the distances between them, are all given from the position of MN: put, therefore, $AQ = a$, $BR = b$, $CS = c$, $DT = d$, $EV = e$, $QR = r$, $QS = s$, $QT = t$, &c. Also put $QP = x$, and the given sum of all the proposed squares $= k^2$.

Then, RP being $= x - r$, $SP = x - s$, &c.

We shall have,
$$\left. \begin{aligned} \overline{AP}^2 &= aa + xx \\ \overline{BP}^2 &= bb + xx - 2rx + rr \\ \overline{CP}^2 &= cc + xx - 2sx + ss \\ \overline{DP}^2 &= dd + xx - 2tx + tt \end{aligned} \right\} = k^2.$$
 by Elem. 8. 2.

Put $aa + bb + cc + dd$ &c. $= \alpha^2$, $r + s + t$ &c. $= \beta$, and $rr + ss + tt$ &c. $= \gamma^2$; and let the number of the given points be denoted by n :

then our equation will be reduced to

$$\alpha^2 + nx^2 - 2\beta x + \gamma^2 = k^2:$$

from whence x is found $= \frac{\beta}{n} + \sqrt{\frac{k^2 - \alpha^2 - \gamma^2}{n} + \frac{\beta^2}{n^2}}$.

Geometrically.

If PQ be drawn to bisect the distance AB, then will $\overline{AP}^2 + \overline{BP}^2 = 2AQ \times BQ + 2\overline{QP}^2$, by the lemma.

And,

And, if QC be drawn, and QR be taken equal to $\frac{1}{3}$ thereof; then will $2\overline{QP}^2 + \overline{CP}^2 = 3QR \times CR + 3\overline{RP}^2$, by the same. Whence, by adding these equal quantities together, and taking $2\overline{QP}^2$ (common) away, we have $\overline{AP}^2 + \overline{BP}^2 + \overline{CP}^2 = 2AQ \times BQ + 3QR \times CR + 3\overline{RP}^2$.

Again, by drawing RD, and taking RS = $\frac{1}{4}$ thereof, we have $3\overline{RP}^2 + \overline{DP}^2 = 4RS \times DS + 4\overline{SP}^2$, by the lemma.

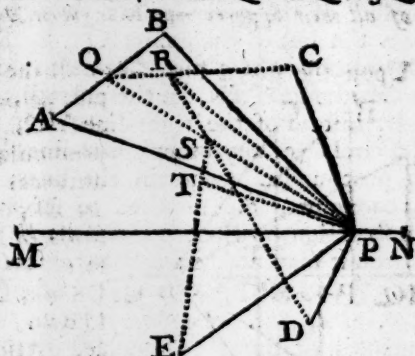
And therefore $\overline{AP}^2 + \overline{BP}^2 + \overline{CP}^2 + \overline{DP}^2 = 2AB \times BQ + 3QR \times CR + 4RS \times DS + 4\overline{SP}^2$, by the addition of equal quantities.

In the same manner, if SE be drawn, and ST be taken = $\frac{1}{5}$ thereof, we shall have $4\overline{SP}^2 + \overline{EP}^2 = 5ST \times ET + 5\overline{TP}^2$. Whence, again, by the addition of equal quantities, $\overline{AP}^2 + \overline{BP}^2 + \overline{CP}^2 + \overline{DP}^2 + \overline{EP}^2 = 2AQ \times BQ + 3QR \times CR + 4RS \times DS + 5ST \times ET + 5\overline{TP}^2$.

Hence it is evident (without proceeding further) that, let the number of the given points be what it will, the square of the last of the lines QP, RP, SP, TP, &c. drawn as above, will always be a given quantity; because the sum of all, the rectangles $2AQ \times BQ$, $3QR \times CR$, $4RS \times DS$, &c. (or that of their equals $AB \times QB$, $QC \times RC$, $RD \times SD$, &c.) is given, by the position of the points A, B, C, D, &c.

A a

In



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In order, therefore, to the geometrical construction let a rectangle be constituted (*by Elem. 7. 6.*) equal to the excess of the given sum of the squares above the sum of the said rectangles: then find a mean proportional between the length of the rectangle, so determined, and that part of its breadth defined by the number of the given points; with which, as a radius, from the last of the points Q, R, S, T, &c. let the circumference of a circle be described; which will intersect the line MN, in the point P, required.

If, instead of the right-line MN, the circumference of a circle (or any other curve-line) given by position, be propounded (wherein the lines, drawn from the given points, are required to meet), the method of construction, will, it is manifest, be exactly the same.

And it may not be amiss to observe farther, that if $a \times AP^2 + b \times BP^2 + c \times CP^2 + d \times DP^2$, &c. (where a, b, c, d , &c. represent known quantities, either numbers or lines) be supposed given, the locus of the point P will, *still*, be the circumference of a circle, determined after the same manner, from the premised lemma.

For, by making in this case $\begin{cases} AQ : BQ :: b : a \\ QR : RC :: c : a+b \\ RS : DS :: d : a+b+c \\ ST : ET :: e : a+b+c+d \end{cases}$

and then proceeding as above, it will appear that $a \times AP^2 + b \times BP^2 + c \times CP^2 + \&c.$ is equal to $a+b \times AQ \times QB + a+b+c \times QR \times RC + a+b+c+d \times RS \times SD + a+b+c+d+e \times ST \times TE + a+b+c+d+e \times TP^2$. Whence it is evident that TP is a given quantity.

PART

PART III.

WHEREIN

The THEORY of GUNNERY, or the
MOTION of PROJECTILES, is considered.

IT is, usually, taken for granted, by *those* who treat of the motion of projectiles, that the force of gravity near the earth's surface is every where the same, and acts in parallel directions; and that the effect of the air's resistance upon very heavy bodies, such as bombs and cannon balls, is too small to be taken into consideration.

That the error arising from the supposition of gravity acting uniformly, and in parallel lines, must be exceeding small, is very obvious; because, even, the greatest distance of a projectile above the surface of the earth, is inconsiderable in comparison of its distance from the center, to which the gravitation tends: but then, on the other hand, it is very certain, that the resistance of the air, to very swift motions, is much greater than it has been commonly represented.

Nevertheless, if the amplitude of the projection, answering to one given elevation, be first found by experiment (which our method supposes to be done) the amplitudes in all other cases, where the elevations and velocities do not very much differ from the first, may be determined, to a sufficient degree of exactness, from the foregoing hypothesis: because, in all such cases, the effects of the resistance will be nearly as the amplitudes *themselves*; and, were they *accurately* so, the proportions of the

amplitudes, at different elevations, would then be the very same as *in vacuo*.

For this reason, and to avoid having recourse to principles and calculations no ways adequate to the experience and understanding of beginners, for whose use this little tract is chiefly intended, I shall, in what follows, conform to the method of other writers, so far, as to take no notice of the air's resistance; but consider the motions as performed *in vacuo*.

Now, in order to form a clear idea of the subject here proposed, the path of every projectile is to be considered as depending on two different forces; that is to say, on the impellant force, whereby the motion is first began (and would be continued, in a right-line), and on the force of gravity, by which the projectile, during the whole time of its flight, is continually urged downwards, and made to deviate more and more from its first direction.

As whatever relates to the track and flight of a ball (neglecting the resistance of the air) is to be determined from the action of these two forces, it will be proper, before we proceed to consider their joint effect, to premise something concerning the nature of the motion produced by each, when supposed to act alone, independant of the other; to which end the two first, of the four following *lemmas*, are premised.

LEMMA

LEMMA I.

Every Body, after the impressed Force, whereby it is put in Motion, ceases to act, continues to move uniformly in a Right-line; unless it be interrupted by some other Force or Impediment.

This is a law of nature, and has its demonstration from experience and matter of fact.

COROLLARY.

It follows from hence that a ball, after leaving the mouth of the piece, would continue to move along the line of its first direction, and describe spaces therein proportional to the times of their description; were it not for the action of gravity; whereby the direction is changed, and the motion interrupted.

LEMMA II.

The Motion, or Velocity, acquired by a Ball, in freely descending from Rest, by the Force of an uniform Gravity, is as the Time of the Descent; and the Space fallen through, as the Square of that Time.

The first part of the lemma is extremely obvious: for, since every motion is proportional to the force whereby it is generated, that generated by the force of an uniform gravity must be as the time of the descent; because the whole effort of such a force is proportional to the time of its action; that is, as the time of the descent.

To demonstrate that the distances descended are proportional to the squares of the times, let the time of falling through any proposed distance AB be represented by the right-line PQ; which conceive to be divided into an indefinite number of very small, equal, particles, represented, each, by the *symbol* m ; and let the distance descended in the first of them be Ac ; in the second cd ; in the third de ; and so on.

Then the velocity acquired being always as the time from the beginning of the descent, it will at the middle of the first of the said particles

B be represented by $\frac{1}{2}m$; at the middle of the second, by $1\frac{1}{2}m$; at the middle of the third, by $2\frac{1}{2}m$, &c. Which values constitute the series $\frac{m}{2}, \frac{3m}{2}, \frac{5m}{2}, \frac{7m}{2}, \frac{9m}{2}$, &c.

But, since the velocity, at the middle of any one of the said particles of time, is an exact mean between the velocities at the two extremes thereof, the corresponding particle of the distance AB may be therefore considered as described with that mean velocity: and so, the spaces $Ac, cd, de, \&c.$ being, respectively, equal to the abovementioned quantities $\frac{m}{2}, \frac{3m}{2}, \frac{5m}{2}$,

$\frac{7m}{2}$, &c. it follows, by the continual addition of these, that the spaces $Ac, Ad, Ae, Af, \&c.$ fallen through from the beginning, will be expressed by $\frac{m}{2}, \frac{4m}{2}, \frac{9m}{2}, \frac{16m}{2}, \frac{25m}{2}$, &c. Which are, evidently, to one another in proportion, as 1, 4, 9, 16, 25, &c. that is, as the squares of the times. Q. E. D.

COROL.

COROLLARY.

Seeing the velocity acquired in any number (n) of the aforesaid, equal, particles of time (measured by the space that would be described in one single particle) is represented by n times m , or $n m$, it will therefore be, as 1 particle of time, is to n such particles, so is $n m$, the said distance answering to the former time, to the distance, $n^2 m$, corresponding to the latter, with the same celerity, acquired at the end of the said n particles. Whence it appears that the space $\frac{n^2 m}{2}$ (found above) through which the ball falls, in any given time n , is just the half of that ($n^2 m$) which might be uniformly described with the last, or greatest, celerity, in the same time.

SCHOLIUM.

It is found, by experiment, that any heavy body, near the earth's surface (where the force of gravity may be considered as uniform) descends about 16 feet, from rest, in the first second of time.

Therefore, as the distances fallen through are proved above to be, in proportion, as the squares of the times; it follows that, as the square of 1 second, is to the square of any given number of seconds, so is 16 feet, to the number of feet a heavy body will freely descend in the said given number of seconds. Whence the number of feet descended in any given time will be found, by multiplying the square of the number of seconds by 16.

Thus the distance descended in 2, 3, 4, 5, &c. seconds, will appear to be 64, 144, 256, 400 F, &c. respectively.

Moreover, from hence, the time of the descent through any given distance will be obtained, by dividing the said distance, in feet, by 16, and extracting the square root of the quotient; or, which comes to the same

same thing, by extracting the square root of the whole distance, and then taking $\frac{1}{2}$ of that root for the number of seconds required: Thus, if the distance be supposed 2640 feet; then, by either of the two ways, the time of the descent will come out 12.84 seconds, or 12". 50^m. 4^m.

It appears also (*from the corol.*) that the velocity per second (in feet), at the end of the fall, will be determined, by multiplying the number of seconds in the fall by 32: thus it is found that a ball, at the end of 10 seconds, has acquired a velocity of 320 feet per second.—After the same manner, by having any two of the four following quantities, *viz.* the force, the time, the velocity, and distance, the other two may be determined: but this not being absolutely necessary in what follows (though equally useful in other disquisitions) I shall put down the several rules, or equations below, in a note,* to be taken, or omitted, at pleasure.

LEMMA

* Let the space freely descended by a ball, in the first second of time (which is as the accelerating force) be denoted by f , also let t denote the number of seconds wherein any distance, d , is descended; and let v be the velocity, per second, at the end of the descent: then will

$$v = 2ft = 2\sqrt{fd} = \frac{2d}{t}$$

$$t = \sqrt{\frac{d}{f}} = \frac{v}{2f} = \frac{2d}{v}$$

$$d = ft = \frac{vv}{4f} = \frac{tv}{2}$$

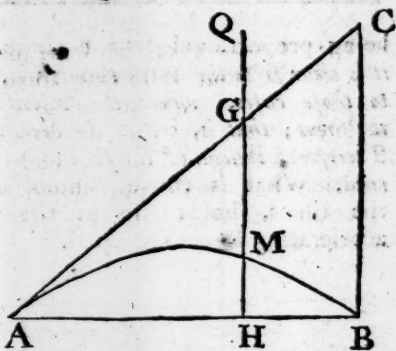
$$f = \frac{d}{t} = \frac{v}{2t} = \frac{vv}{4d}$$

All which equations are very easily deduced from the two original ones, $d = ft$, and $v = 2ft$, already demonstrated; the former in the proposition itself; and the latter, in the corollary to it; by which it appears that the measure

LEMMA III.

A Ball, projected in the Direction of a Right-line AC making an Angle with the Horizon AB, is, by Means of its Gravity, deflected continually, more and more, from its first Direction; but the Celerity with which it approaches any Perpendicular (BC) to the Horizon, is neither increased nor decreased by the Action of Gravity.

For, let a line HMQ , perpendicular to the horizon, be conceived to move parallel to itself, towards BC , along with the ball: then, as the gravity always acts in this line, it can have no effect in making it, either, move faster or slower towards BC ; but is wholly employed in drawing down the projectile along the same, from its first direction AC ; and thereby causing it to describe a curve-line AMB .



COROL.

measure of the velocity at the end of the first second is $2f$; whence the velocity (v) at the end of t seconds must consequently be expressed by $2f \times t$ or $2ft$.

Having proceeded thus far, I shall here take the opportunity to point out, and rectify an inadvertant expression, at p. 230 of my Book of Fluxions, relating to this subject.

—It is there said, by way of remark, that, whatever ratio the times have with respect to the distances descended, &c. the same also will the velocities have,

B b

being

COROLLARY.

Hence it appears that the projectile, at the end of a given time, is in the very same vertical line HG, as it would be in, if gravity was not to act; and that the horizontal distance AH, as well as the hypotenuse AG, is proportional to the time wherein the projectile actually moves through the arch AM, corresponding to the said distance.

being proportional to the times. *Which observation, it is clear from the words themselves, ought to be restrained to those cases, where the velocities are proportional to the times; that is, where the accelerating forces are equal. Therefore, instead of the said observation, as it now stands, read, What is above demonstrated with respect to the times, holds also in the velocities, when the accelerating forces are equal.*

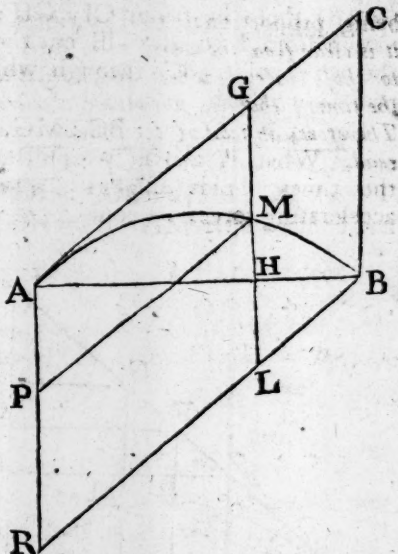
LEMMA

LEMMA IV.

The Deflections arising from the Action of Gravity, or the vertical Distances, MG, BC, intercepted by the Path of the Ball and the Line of Direction, are in proportion, to one another, as the Squares of the corresponding Parts AG, and AC, of the said Line of Direction.

Conceive a line GML to be carried along, parallel to BC, so that its extreme point G may trace out the line of direction AC, in the very same manner as the projectile itself, were it not to be deflected therefrom by the action of gravity: then,

since, by the preceding lemma, the projectile is always in the line GML, and the force of gravity is, wholly, employed in urging it downwards, along that line; the effect produced by the said force, or the distance MG of the ball from the extreme point G, at the end of any given time, will consequently be the very same, as if the line GML (instead of being carried uniformly towards CB) was to have continued in its first position AR, and the ball suffered to descend from rest along that line; the force employed being the same in both cases. But it is proved, in Lemma 2, that the spaces AP and AR that would be described in descending, freely,



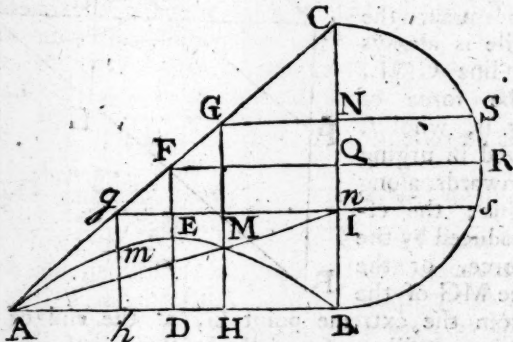
from

from rest, are as the squares of the times; therefore their equals GM and CB are likewise as the squares of the same times; that is, of the times wherein AM and AMB are described: which are to each other as $\overline{AG}^2$ to $\overline{AC}^2$ (by Corol. to Lem. 3.)

COROLLARY I.

If B be taken as the point where the projectile impinges on the horizon AB, and in BC (perpendicular to AB) there be taken $CI=GH$; then a right-line, drawn from I to A, will cut the vertical line HG in the very point (M) through which the center of the projectile passeth.

For $\overline{AC}^2 : \overline{AG}^2 :: BC : MG$, as above.
 and $\overline{AC}^2 : \overline{AG}^2 :: \overline{BC}^2 : \overline{HG}^2$, by *sim. triangles*.
 Theref. $BC : MG :: \overline{BC}^2 : \overline{HG}^2$, by *equality of ratios* :
 and consequently $MG \times BC = \overline{HG}^2$.



This, turned into an analogy, gives $BC : HG :: HG : MG$; and so likewise is BC to CI (because of the parallel lines.)⁶⁰ Whence it is evident that HG and CI are equal to each other, in all positions of HG : by means whereof, as many points in the curve, as you please, may be determined.

COROL.

$$\infty. bc:hg::(ab:ah::) bc:hm$$

$$bc:(bc-bi)cl::hg:(hg-hm):mg \therefore cl=mg$$

COROLLARY II.

If GNS be drawn, parallel to AB, intersecting BC in N, and meeting the circumference of a semi-circle, described upon BC, in S; it will further appear, that the height (HM) of the ball is everywhere a third proportional to BC and NS. For, since $BC : HG :: HG : MG$, it follows, by division, that $BC : CN :: HG : (BN) : HM = \frac{CN \times BN}{BC} = \frac{NS^2}{BC}$. *(Elem. 16. 3.)* Hence it is evident that the projectile will be at its greatest height, when it has performed, just, the half of its flight; since it is well known that

NS (and consequently $\frac{NS^2}{BC}$) is the greatest possible when it coincides with the radius QR.

It appears moreover that the heights *hm*, HM, of the ball, in any two vertical lines, equally distant from that passing through the highest point E of the trajectory, are equal; because the corresponding ordinates *ns* and NS are equal, as being equally distant from the center Q of the semi-circle.—Lastly, it is apparent that the greatest altitude DE is $= \frac{1}{4}BC$; because $\frac{NS^2}{BC}$, when

NS coincides with QR, becomes $= \frac{\frac{1}{4}BC^2}{BC} = \frac{1}{4}BC$.

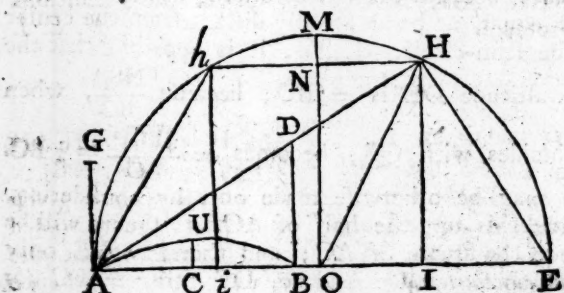
Which may be otherwise made out, by considering, that, as AF is but the half of AC, its square will be only $\frac{1}{4}$ of the square of AC; and therefore FE only $\frac{1}{4}$ of BC. (by the lemma) And so, DF being $= \frac{1}{4}BC$ (by sim. triang.) DE, as well as FE, must be $= \frac{1}{4}BC$.

PROPOSITION

PROPOSITION I.

The Amplitude, or horizontal Range of a Piece, with a given Charge of Powder, at an Elevation of 45 Degrees, being known, from Experiment; to determine the Elevation so as to hit an Object, at a given Distance, on the plane of the Horizon; the Quantity of Powder remaining the same.

Let PQ be the given amplitude, at an elevation (QPR) of 45 degrees; let AB be the given distance of the object, and BAD the required elevation: in AB, produced, take AO=PQ, with which as a radius, from the center O, let a semi-circle AME be described, and let AD, produced, meet the periphery thereof in H; join E, H, and make DB and HI perpendicular to AE.



Since the charge of powder, or the velocity at both the elevations QPR and BAD, is supposed to be the same, the times of flight, during which the distances PR and AD would be uniformly described with that velocity, will therefore be to each other, directly as the said distances; and consequently $PR^2 : AD^2 :: RQ$

RQ : DB (by principles already explained, vid. Lem. 3 and 4.)

But $\overline{PR}^2 (= \overline{PQ}^2 + \overline{RQ}^2) = 2\overline{RQ}^2$ (because the angle P ($=45^\circ$) $=R$.) Hence the above proportion becomes $2\overline{RQ}^2 : \overline{AD}^2 :: RQ : DB$; from which we have $\overline{AD}^2 = 2RQ \times DB = AE \times DB$, by construction.

Moreover, from the similar triangles ABD and AHE, we have $AE : EH :: AD : DB$; whence $EH \times AD = AE \times DB = \overline{AD}^2$ (p. above) and consequently $EH = AD$: and so, the triangles EHI and ADB being equiangular, it is plain that HI is also equal to AB. Whence follows this easy

Construction.

With an interval equal to the given amplitude, at the elevation of 45° , let a semi-circle AME be described; make OM' perpendicular to the diameter thereof, in which take ON equal to the given distance AB, and through N, parallel to AE, draw HN, intersecting the circumference in H and b; then either of the directions AH, or Ab, will answer the conditions of the problem.

COROLLARY I.

If OH be drawn, the angle EOH will be $=2EAH$ (Elem. 9. 3.) and it will be, as OH (PQ) : HI. (AB) :: radius : sine EOH; that is, in words, as the given amplitude, at 45° elevation, is to any other proposed amplitude, so is the radius, to the sine of twice the elevation corresponding to the latter. From whence it is evident, that the horizontal amplitudes, at different elevations, are to one another as the sines of the doubles of the said elevations; and that, the amplitude of the projection at an elevation of 45 degrees (when HI coincides with MO) is the greatest possible.

COROL.

COROLLARY II.

Since it is found above that $AE \times DB (= \overline{AD})^2 = \overline{EH}^2$, it follows, that $\overline{AE}^2 : \overline{EH}^2 :: \overline{AE}^2 : AE \times DB :: AE : DB :: \frac{1}{2}AO (\frac{1}{2}AE) : CV (\frac{1}{2}DB. \text{ vid. Corol. 2. to Lem. 4.})$ that is, as the square of the *radius*, is to the square of the *sine* of the angle of elevation, so is half the greatest horizontal amplitude, to the greatest altitude of the projectile.

Cor. 3

Hence it appears, that the distance which the ball would ascend, if projected in a vertical direction (*usually called the impetus*) is just one half of the greatest amplitude; since, in this case, the *sine* of the elevation becomes equal to the *radius*.

Cor. 4

Therefore, as a body (*in vacuo*) ascends and descends with the same celerity; and seeing the distance AG, expressing the perpendicular ascent, is as the square of the celerity at A (*p. Lem. 2*); it follows that the greatest horizontal amplitude AO, being $= 2AG$, is also as the square of the same celerity.

Cor. 5.

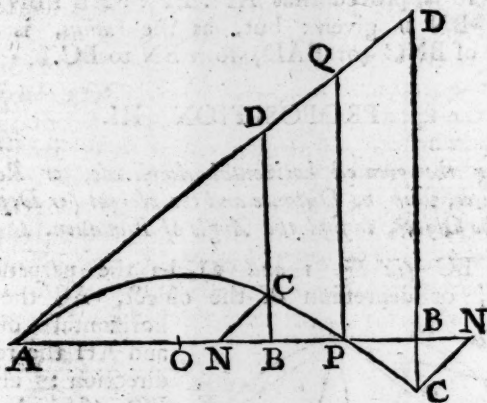
From whence, and *Corol. 1*, it is manifest, that the amplitudes, when both the elevations, and the velocities, differ, will be to each other in a *ratio* compounded of the *ratio* of the *sines* of the double elevations, and the *duplicate ratio* of the velocities.

PROPOSITION

PROPOSITION II.

The greatest horizontal Amplitude, and the Angle of Elevation being given; to find at what Distance the Piece ought to be planted, to hit an Object, whose Height above, or Depression below, the Level of the Piece is also given.

Let BC be the perpendicular height, or depression of the proposed object, and AB the required distance; let BC, produced, meet the line of direction AD in D, and let P be the place where the path of the projectile (produced) meets the level of the piece: make PQ perpendicular to AP, and CN parallel to AD.



By the last proposition it will be, as *radius* : *sin. 2A* :: the greatest horizontal amplitude, to the distance AP; which, therefore, is known.

Moreover, it appears from the fourth proportion, in *Corol. 1.* to *Lem. 4.* that $PQ : BD :: BD : CD$.

But $\left\{ \begin{array}{l} PQ : BD :: AP : AB \\ BD : CD :: AB : AN \end{array} \right\}$ by similar triangles.

Therefore, by equality of ratios.

$AP : AB :: AB : AN$;

C c

whence

whence $AP : PB :: AB : BN$ (by division)
and consequently $AP \times BN = AB \times PB$.

Let AP be now bisected, in O ; then $\underline{AB \times BP}$ (or

its equal $AP \times BN$) being $= \overline{AO}^2 + \overline{OB}^2$ (*Elem.* 7. 2.) *Euclid*

we therefore have $OB^2 = AO^2 + AP \times BN = AO \times AO + 2BN$; whence AB is likewise known. = *Q. E. I.*

$$= 40 + 103 \text{ APX} \cdot BC \div PQ = BN \text{ also } = \cos:$$

COROLLARY.

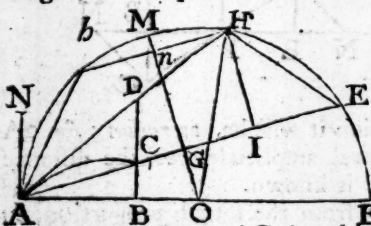
If the amplitude AP , the elevation PAQ , and the distance AB of any perpendicular BCD from the place of projection A , be supposed given; then the height, or depression of the ball in that perpendicular, may, from hence, be found.

For it is proved that $AP : BP :: AB : BN$; from which BN is given: but, as the *radius*, is to the *tangent* of BNC (or BAD) so is BN to BC .

PROPOSITION III.

Having the greatest horizontal Amplitude, or Range of a Piece, with the Distance and the Height (or Depression) of the Object, to find the Angle of Elevation.

Let BC (in fig. 1 and 2) be the perpendicular height, or depression of the object, AB the given



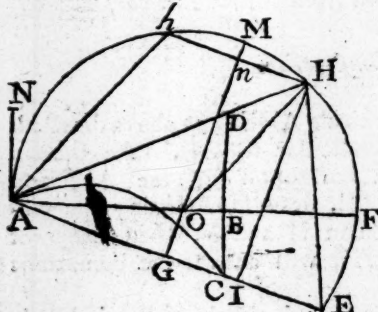
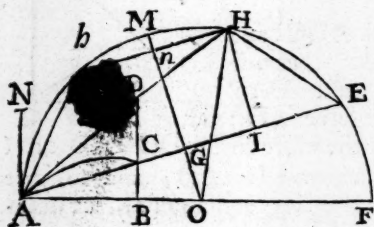
horizontal distance, and AH the required direction: also let PQ (*fig. 3.*) be the greatest horizontal amplitude, answering to 45° of elevation, (*vid. Corol.*)

1. *Prop. 1.* Draw AC, in which produced (if need be) take $AG = PQ$; make MGO perpendicular to AG, meeting AB produced (if necessary) in O; and from the center O, with the interval OA, let the circumference of a circle be described, intersecting AG produced,

Then, as $AG (PQ) : On :: \text{co-fine of } OAG : \text{co-fine } HO_n = (HAb)$ the difference of the two required elevations : from which the elevations *themselves* will be known : and from whence, if BC be supposed to vanish, the solution to *Prop. 1.* will be obtained ; being only a particular case of that above. Q. E. I.

COROLLARY I.

Hence the time of the flight may likewise be determined: for, $S. AHE$ (ΔCD) $\propto S. HAE :: AE : (\frac{S.HAE}{S.AHE} \times AE =) HE = AD$ (*p. above*) which being proportional to the time of the flight (*vid. Lem. 3.*) it follows that the said time will, always, be as $\frac{S.HAE}{S.ACD}$; because AE is supposed constant. There-



of the perpendicular ascent, or descent, is to the true time of the flight.

COROL.

COROLLARY II.

If the elevation of the piece, together with the distance, and the height (or depression) of the object, be given, the *impetus* may, from hence, be also found.

For, first, it will be $AB : BC :: \text{radius} : \text{tang. BAC}$; whence (as BAD is given) EAH will likewise be known.

Then, $S. EAH (CAD) : S. AHE (ACD) :: HE (AD) : AE$. Also $S. ADC : \text{radius} :: AB : AD$.

Therefore, by compounding these proportions, we have $S. CAD \times S. ADC : \text{radius} \times S. ACD :: AB : AE$; which is four times the required *impetus* (*vid. Corol. 3. to Prop. 1.*)

(i.e. by just
Rad. $\times$ A
 $\frac{S. ADC}{\text{for AD}}$)

COROLLARY III.

Moreover, if the elevation, and the *impetus* be given, the amplitude of the projection on an ascending, or descending plane ACE, whose inclination is given, may from hence be derived.

For, $S. AHE (ACD) : S. EAH (CAD) :: AE : EH (AD)$. And $S. ACD : S. ADC :: AD : AC$.

By the composition of which proportions we have $\square S. ACD : S. CAD \times S. ADC :: AE : AC$; whence AC is given.

COROLLARY IV.

Hence, also, may the ratio of the amplitudes, on the same plane, at different elevations, be deduced: for the first and third terms, of the last proportion, continuing invariable, the ratio of the 2^d and 4th will likewise be invariable; that is, the rectangle under the *sine* of the elevation above the plane, and the *co-sine* of the elevation above the horizon, in any one case, will be to the like rectangle, in any other, as the amplitude in the former case, to the amplitude in the latter.

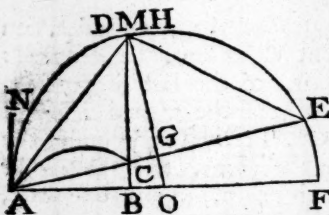
COROL.

COROLLARY V.

But, if the elevation be supposed constant; and the plane's inclination to vary; then, since, by the above proportion, AC is *universally*
$$= \frac{S. CAD \times S. ADC}{\square \text{ sine } ACD} \times AE$$
 (where AE and the angle ADC are supposed constant) it follows that the amplitude will be, barely, as $\frac{S. CAD}{\square S. ACD}$; that is, *inversely*, as the square of the *co-sine* of the inclination of the plane, applied to the *sine* of the elevation above the plane.—If both the inclinations, and the elevations, differ, it will appear, from the same equation, that, the amplitude will be, *universally*, as the rectangle of the *sine* of the elevation above the plane, and the *co-sine* of the elevation above the horizon, applied to the square of the *co-sine* of the plane's inclination.

COROLLARY VI.

Since it is proved that HI is, always, equal to AB, it is evident that, when the former coincides with MG, and thereby becomes a *maximum*, the latter will also be a *maximum*: in which circumstance AC will likewise be a *maximum*; and the point D will then coincide with M and H (as in the annexed figures) because AD and EH are always equal to each other.



Therefore, since, in this case, the angle HAE is ($= E$) $= NAH$, it appears that the amplitude, on any inclined plane, will be the greatest possible when the line of direction AH bisects the angle EAN, included between the plane and the zenith.

CORCL.

BAC will likewise be known; and from thence, the elevation, BAH, *by Corol. 6.*

From the latter of the two cases here considered (where the object is supposed below the level of the piece, *as in fig. 2.*) the greatest amplitude of a ball, projected from a given height above the plane of the horizon, is given; being $= \sqrt{AG \times AG + 2BC}$, (*as above*) in which BC denotes the given height.

COROLLARY X.

But, if the horizontal distance AB is given, and it be required to determine the greatest height the ball can possibly reach, in the perpendicular BCD; we shall then have, $HG (AB) : AG :: \text{radius} : \text{tang. of the elevation } (BAH \text{ or } AHG)$; and, as $\text{radius} : \text{tang. } BAC (= 2BAH \propto 90) :: AB : BC$; which therefore is known. But, because $AC = AG + BC$, and $BC^2 = AC^2 - AB^2$, the value of BC will, *also*, be truly expressed by $\frac{AG^2 \propto AB^2}{2AG}$.

COROLLARY XI.

Lastly, if there be given the perpendicular height or depression, of the object, and its horizontal distance, in order to determine the elevation, and the *least* impetus, to hit the object: then it will be as $AB : BC :: \text{radius} : \text{tang. } BAC$; whence the elevation BAH is also known, *by Corol. 6*: and, as $\text{radius} : \text{tang. } AHG (BAH) :: HG (AB) : AG$; the half of which is the impetus, *by Prop. 1. Corol. 2.*

Here

Here follow the practical Solutions of the several Cases depending on the foregoing THEORY.

I. Of Projections made on the Plane of the Horizon.

PROBLEM I.

The greatest Amplitude of a Piece being known (from Experiment) to find the Amplitude, at any proposed Elevation.

SOLUTION.

As the radius is to the sine of double the proposed elevation, so is the given, to the required, amplitude (by Prop. I. Cor. I.)

Ex. Let the gr. amp. be 8000 feet, and the given elev. $30^{\circ} 16'$.

Then, as radius	.	.	.	10. 0000
to sine $60^{\circ} 32'$	.	.	.	9. 9398
so is 8000	.	.	.	3. 9030
Amp. req. 6965 f.	.	.	.	3. 8428

PROBLEM II.

The Impetus, or the greatest Amplitude (which is the Double thereof) being known, to find the Elevation, to strike an Object at a given Distance.

SOLUTION.

As the greatest amplitude, is to the given distance; so is the radius, to the sine of the double elevation (by Prop. I. Corol. I.)

Ex. Let the gr. amp. be 7500 f. and the given distance 5620 f.

D d

Then,

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Then, as 7500	.	.	.	.	3. 8750
is to 5620	.	.	.	.	3. 7497
so is radius	.	.	.	.	10. 0000
to the sine of $48^{\circ} 32'$, or $131^{\circ} 28'$					9. 8747

Therefore $24^{\circ} 16'$ is the lower, and $65^{\circ} 44'$ the higher elevation, required.

PROBLEM III.

The Angle of Elevation, and the Distance of an Object, being given, to find the Impetus, so as to strike the Object.

SOLUTION.

As the sine of twice the elevation, is to the radius, so is the distance of the object to twice the impetus (by Prop. 1.)

Ex. Let the elev. be $32^{\circ} 12'$ and the given dist. 6500 f.

Then, as sine $64^{\circ} 24'$	.	.	.	.	9. 9551
is to radius	.	.	.	.	10. 0000
so is . 6500	.	.	.	.	3. 8129
to . . 7208	.	.	.	.	3. 8578

Whence 3604 is the impetus required.

PROBLEM IV.

The Amplitude, at any one known Elevation being given, to find the Amplitude at any other known Elevation.

SOLUTION.

As the sine of double the first elevation, is to the sine of double the second, so is the amplitude at the former, to that at the latter (by Prop. 1.)

Ex. Let the first elev. be $25^{\circ} 12'$, the second $36^{\circ} 15'$, and the given amp. 5250 f.

Then,

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Then, as sine $50^{\circ} 24'$	. co. ar.	0. 1132
to sine . $72^{\circ} 30'$	. .	9. 9794
so is . . 5250 f.	. .	3. 7201
to the amp. req. 6498 f.	. .	13. 8127

PROBLEM V.

The Amplitude of the Projection, with a given Quantity of Powder, being known; to find the Requisite of Powder, so as to strike an Object at a given Distance; the Elevation remaining the same.

SOLUTION.

As the given amplitude, is to the proposed distance, so is the given weight, or quantity of powder, to the quantity sought, nearly. *

Ex. Suppose the requisite of powder to throw a shot 4000 feet, at 45° elevation, to be 16lb. What quantity is necessary, to strike an object at the distance of 5000 feet.

Here it will be, as $4000 : 5000 :: 16 : 20$ lb. the quantity sought.

* In this solution, the velocity communicated to the ball, is supposed to be in the subduplicate ratio of the quantity of powder; which is not strictly true, especially in large charges; since a considerable part of the powder, in such cases, is blown out, unfired: there are, besides, other reasons to be assigned, why, the velocity cannot be exactly in the proportion above specified.

PROBLEM VI.

The Distance of the Object, and the Elevation of the Piece, being given; to determine the Time of the Flight.

SOLUTION.

As the radius is to the tangent of the elevation, so is the given distance of the object, *in feet*, to the square of four times the number of *seconds*, required (*vid. p. 184 and 187.*)

Ex. Let the elevation be 32° , and the distance of the object 5280 feet.

Then, radius . . .	10. 0000
tang. 32° . . .	9. 7958
dist. 5280 . . .	3. 7226
	13. 5184
$57^\circ 44'$. . .	1. 7592

$\frac{1}{4}$ of which, or $14', 36''$, is the time required.

But, when the elevation is 45 degrees (which is commonly the case, in throwing bombs, &c.) then $\frac{1}{4}$ of the square root of the distance, *in feet*, will give the number of seconds taken up in the flight. The knowing of which will be of use in adjusting the fuse.

II. *Of Projections, when the Object fired at, is above, or below, the Level of the Piece.*

PROBLEM VII.

The horizontal Distance, and the Angle of Elevation, or Depression of an Object, being given, with the Elevation of the Piece; to find the Impetus, so as to hit the Object.

SOLUTION.

As the rectangle of the sine of the elevation of the piece above the object, and the co-sine of its elevation

vation above the horizon, is to the rectangle under the radius and the co-sine of the object's elevation, or depression; so is $\frac{1}{2}$ of the given horizontal distance of the object, to the impetus required. (*by Prop 3. Corol. 2.*)

Ex. Let the horizontal distance of the object be 5600 feet, and its elevation $8^{\circ} 15'$; and let the elevation of the piece be $32^{\circ} 30'$: then

$24^{\circ} 15'$, co-ar. of its sine	0. 3865	} to be added
$32^{\circ} 30'$, co-ar. of its co-s.	0. 0740	
radius	10. 0000	
$8^{\circ} 15'$, its co-sine	9. 9955	
1400 f.	3. 1461	
4000 f. the imp. req.	<u>23. 6021</u>	

PROBLEM VIII.

The horizontal Distance, and the Angle of Elevation, or Depression of an Object, being given, together with the Impetus; to find the Elevation of the Piece, to hit the Object.

SOLUTION.

As the radius is to the tangent of the object's elevation, or depression, so is twice the impetus to a fourth number; which add to, or subtract from, the given horizontal distance, according as the object is elevated, or depressed: then say,

As twice the impetus is to the sum, or remainder, so is the co-sine of the given elevation, or depression, to the co-sine of an angle; which added to, and subtracted from, the angle included between the object and the zenith (or vertical point), gives the double of the complements of two different elevations, whereby the ball may hit the object (*see Prop. 3.*)

Ex. Let the horizontal distance of the object be 5600 feet, and its elevation $8^{\circ} 15'$; and let the given impetus be 4000 f.

Then,

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Then, as radius	10. 0000
is to tang. $8^{\circ} 15'$	9. 1613
so is 8000	3. 9031
to . 1160	<u>13. 0644</u>

This added to 5600 gives 6760.

Therefore, as 8000	3. 9031
is to . 6760	3. 8299
so is co-f. $8^{\circ} 15'$	9. 9955
to the co-f. $33. 15$	<u>9. 9223</u>

Which, added to, and subtracted from $81^{\circ} 45'$, gives $115^{\circ} 00'$, and $48^{\circ} 30'$, respectively: the halves of which are $57^{\circ} 30'$, and $24^{\circ} 15'$; whose complements $32^{\circ} 30'$, and $65^{\circ} 45'$, are the two elevations required.

PROBLEM IX.

The Impetus, and the Angle of Elevation, being given; to find at what Distance the Piece ought to be planted, to hit an Object, whose Distance above, or below, the Level of the Piece, is also given.

SOLUTION.

As the radius, is to the sine of twice the given elevation, so is the impetus, to half the horizontal amplitude, at that elevation (*by Prop. 1.*)

And, as the radius, is to the co-tangent of the elevation, so is twice the perpendicular height, or depression, of the object, to a fourth-proportional; which take from, or add to, half the horizontal amplitude, according as the object is elevated, or depressed; then find a mean proportion between the half-amplitude, and the sum, or remainder; which, added to the said half-amplitude, gives the distance sought (*by Prop. 2.*)

Ex. Let the impetus be 3000 feet, the elevation 40° , and the height of the object 200 feet.

Then,

Then, as radius	.	.	.	10. 0000
to sine . 80°	.	.	.	9. 9933
so is . 3000	.	.	.	3. 4771
to . . 2954	.	.	.	1 3. 4704
And, as radius	.	.	.	10. 0000
is to co-t. 40°	.	.	.	10. 0762
so is . 400	.	.	.	2. 6021
to . . 477	.	.	.	1 2. 6783

Now, the difference between the two numbers above found is 2477; whose log. being added to that of 2954, and the sum divided by 2, the quotient will be the log. of 2705, the required mean proportional: whence the distance fought comes out 5659 feet.

III. Of Projections on Planes, inclined to that of the Horizon.

PROBLEM X.

The Inclination of the Plane, and the Elevation and Impetus of the Piece, being known; to find the Amplitude of the Projection.

SOLUTION.

As the square of the co-sine of the plane's inclination to the horizon, is to the rectangle of the sine of the elevation above the plane and the co-sine of the elevation above the horizon, so is 4 times the impetus, to the amplitude of the projection (*by Prop. 3. Cor. 3.*)

Ex. 1. Let the impetus be 4000 feet, the elevation $32^{\circ} 30'$, and the ascent of the plane $8^{\circ} 15'$. Then the elevation above the plane will be $24^{\circ} 15'$; and the operation as follows.

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8° 15', co-arith. of its co-f.	0. 0045	} to be added
<i>The same repeated</i>	0. 0045	
24° 15', its fine	9. 6135	
32° 30', its co-f.	9. 9260	
16000 f. its log:	4. 2041	
5658 feet the amp. req.	2 3. 7526	

Ex. 2. Let the elevation and impetus be the same as in the last example; but let the plane in this case have a descent of 8° 15' (instead of an equal ascent.)

Then the operation will stand thus,

8° 25', co-ar. of its co-f.	0. 0045	} to be added
<i>The same repeated</i>	0. 0045	
40° 45', its fine	9. 8147	
32° 20', its co-fine	9. 9260	
16000 f. its log.	4. 2041	
8992 f. amp. req.	2 3. 9538	

PROBLEM XI.

The Inclination of the Plane, the Elevation of the Piece, and the Amplitude of the Projection, being given; to find the Impetus.

SOLUTION.

As the rectangle of the sine of the elevation above the plane and the co-sine of the elevation above the horizon, is to the square of the co-sine of the plane's inclination, so is the given amplitude, to 4 times the required impetus (*by Prop. 3. Cor. 3.*)

Ex. Suppose the plane to have an ascent of 8° 15', and that the amplitude thereon, at an elevation of 32° 30', is 5658 feet. Then, the elevation above the plane being 24° 15', we shall have

24° 15' co-ar. of its sine . . .	0. 3865	} to be added
32° 30' co-ar. of its co-f. . .	0. 0740	
8° 15' its co-f.	9. 9955	
The same	9. 9955	
5658 f. its log.	3. 7526	
16000 f.	2 4. 2041	

$\frac{1}{4}$ of which, or 4000 feet, is the impetus req.

PROBLEM XII.

The Inclination of the Plane, the Impetus, and the Amplitude being given, to determine the Elevation.

SOLUTION.

As the radius is to the co-sine of the plane's inclination, so is the given distance, on the plane, to the horizontal distance corresponding: from which, the impetus, and the plane's (or object's) elevation or depression, the elevation of the piece may be found, by *Prob. 8*.

Ex. Let the plane have an ascent of 8° 15', and let the given amplitude thereon be 5658 feet, supposing the *impetus* to be 4000 feet.

Then, as the radius . . .	10. 0000
is to the co-sine of 8° 15' . .	9. 9955
so is 5658 feet . . .	3. 7526
to . 5600 feet . . .	1 3. 7481

the horizontal distance of the place where the ball impinges: as for the rest of the operation, it is exactly the same as in the example to *Prob. 8*; for which reason it will be needless to repeat it here.

PROBLEM XIII.

Having the Angle of Elevation, or Depreffion of the Object, together with the Elevation, and the Impetus of the Piece; to determine the Time of the Flight.

SOLUTION.

As the co-sine of the elevation, or depreffion of the object, is to the sine of the elevation of the piece above the object, so is half the square root of the number of feet in the *impetus* given, to the required number of seconds in the flight.

Ex. Suppose the elevation of the object to be $8^{\circ} 30'$; that of the piece 45° ; and the *impetus* 3600 feet.

Then, the square root of 3600 being 60, we have

As the co-sine $8^{\circ} 30'$. co-ar.	0. 0048
is to the sine $36^{\circ} 30'$. . .	9. 7744
so is 30	<u>1. 4771</u>
to 18,04 seconds req. . . .	11. 2563

IV. *Of the MAXIMA and MINIMA, in the Motion of Projectiles.*

PROBLEM XIV.

The greatest horizontal Amplitude being given, to find the greatest Amplitude on a Plane whose Inclination to the Horizon is also given.

SOLUTION.

Take half the angle included between the plane and the zenith; the complement of which is the required elevation (*by Prop. 3. Cor. 6.*) Then say, as the tangent of the said elevation, is to the tangent of the plane's inclination, so is the given amplitude, to the difference between it and the amplitude fought (*by Prop. 3. Corol. 7.*)

Ex.

Ex. Let the greatest horizontal range be 8000 feet, and the inclination of the plane $12^{\circ} 30'$, descending.

Here the angle included between the plane and the zenith, or vertical point, being $102^{\circ} 30'$, the half thereof will be $51^{\circ} 15'$, and the elevation $38^{\circ} 45'$.

Theref. as tang. $38^{\circ} 45'$	. . .	<u>9. 9045</u>
is to tang. . $12^{\circ} 30'$	. . .	9. 3457
fo is . . . 8000 f.	. . .	<u>3. 9031</u>
to . . . 2210 f.	. . .	3. 3443

Which, added to 8000 (because the plane descends) gives 10210 feet, for the true answer in this case.

PROBLEM XV.

The greatest Amplitude, on an inclined Plane, being given, to find the greatest Amplitude, on the Plane of the Horizon.

SOLUTION.

As the radius, is to the sine of the plane's inclination, so is the given amplitude, to the difference between it and the required amplitude (*by Prop. 3. Corol. 8.*)

Ex. Let the inclination of the plane be $12^{\circ} 30'$, descending, and the given amplitude 10210 feet.

Then, as radius	<u>10. 0000</u>
is to S. $12^{\circ} 30'$	9. 3353
fo is . 10210	<u>4. 0090</u>
to . . 2210	3. 3443

Which taken from 10210, gives 8000 feet, for the true distance sought.

PROBLEM XVI.

The Impetus, and the perpendicular Height, or Depression, of an Object being given, to find the greatest horizontal Distance at which the Ball can, possibly, hit the Object, and also the Elevation answering thereto.

SOLUTION.

Take the difference of the two given quantities, if the object be elevated; but otherwise, their sum; then find a mean proportional between the Impetus and said difference, or sum; the double of which will be the distance sought (*by Prop. 3. Cor. 9.*)

For the elevation, it will be, as the distance thus found is to the height, or depression of the object, so is the radius to the tangent of an angle; which added to, or subtracted from, 90 degrees, respectively, gives the double of the required elevation (*by Prop. 3. Cor. 9.*)

Ex. Let the impetus be 4000 feet, and the depression of the object, below the level of the piece, 2210 feet.

Here we are to take a mean proportional between 4000 and 6210; which is $= \sqrt{4000 \times 6210} = 4984$; whose double 9968 is the required distance.

Moreover, we have, as 9968 . 3. 9986

is to . . . 2210 . 3. 3443

so is radius . . . 10. 0000

to tang. . . 12° 30' . 9. 3457

whence the elevation appears to be 38° 45'

From this problem, the greatest amplitude of a ball, projected from a given height above the level of the horizon, is given.

PROBLEM

PROBLEM XVII.

Having the horizontal Distance of the Object, together with its perpendicular Height, or Depression, above or below the Level of the Piece, to determine the least Impetus, possible, whereby the Ball may reach the Object; and also the Elevation corresponding.

SOLUTION.

As the horizontal distance of the object is to its perpendicular height or depression, so is the radius to the tangent of an angle; which added to, or subtracted from 90 degrees, gives the double of the elevation (by Prop. 3. Cor. 6.)

And, as the radius is to the tangent of the elevation, so is the given horizontal distance, to twice the impetus required (by Prop. 3. Corol. II.)

Ex. Let the horizontal distance be 9968 feet, and the distance of the object below the level of the piece 2210 feet.

Then, as . 9968	.	.	3. 9986
is to . . 2210	.	.	3. 3443
so is rad. . . .	.	.	10. 0000
to tang. $12^{\circ} 30'$	.	.	9. 3457

Therefore the elevation is $38^{\circ} 45'$; and it will be

As radius	.	.	10. 0000
is to tang. $38^{\circ} 45'$	.	.	9. 9945
so is . . 9968	.	.	3. 9986
to . . . 8000	.	.	3. 9031

The half of which, or 4000 feet, is the impetus sought.

PROBLEM

PROBLEM XVIII.

To find the greatest Height a Ball can, possibly, reach in a Perpendicular to the Horizon; the Impetus and the Distance of the Piece from the said Perpendicular, being given.

SOLUTION.

Find a third proportional to the impetus and half the given distance, which subtract from the impetus, and the remainder will be the answer (*by Prop. 3. Corol. 10.*)

Thus, if the impetus be 4000 f. and the given distance 6000 feet, then it will be, as $4000 : 3000 :: 3000 : 2250$; which taken from 4000, leaves 1750 feet, for the greatest height the ball can possibly reach in the proposed perpendicular, with that impetus.

PART IV.

EXHIBITING

A new, and very comprehensive Method for extracting the Roots of Equations in Numbers; by increasing the Dimensions of the unknown Quantity.

IN this method (as in the common method by converging series) the value sought is, first of all, to be nearly estimated (either from the equation itself, or from the nature of the problem whence it is derived); and some unknown quantity (as z) must be assumed to express the difference between that value (which we will denote by r) and the true value required: and then, by substituting $r+z$ instead of its equal, in the given equation, a new equation will emerge, affected only with z and known quantities.

The equation being thus prepared for a solution, multiply it into two, or more succeeding terms of the series $1 + Az + Bz^2 + Cz^3 + Dz^4$ &c. (according to the degree of exactness necessary) and let the coefficients of the homologous terms of the new equation hence arising, wherein the square, and the next, higher powers of z are concerned, be made equal to nothing: by which means the value of A , &c. will be determined, and as many terms of the equation destroyed, at the same time, as there are terms in the assumed multiplicator *minus* one. And the terms involving the superior powers, which yet remain, being of very small value, may also be rejected: whence the equation will be reduced to a simple one: from which the value of z will be found.

Let

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Ex. 1. Let the equation propounded be $x^2=50$.

Here, x being something greater than 7, put $7+z=x$; then the given equation will become $49+14z+z^2=50$;

or, $-1+14z+z^2=0$. This, multiplied by $1+Az+Bz^2$,

$$\text{gives } \left\{ \begin{array}{ccc} -1+14z+z^2 & * & * \\ * - Az + 14Az^2 + Az^3 & * & * \\ * & * - Bz^2 + 14Bz^3 + Bz^4 & * \end{array} \right\} = 0.$$

Whence, by equating the coefficients of the like-terms, we have $1+14A-B=0$, and $A+14B=0$.

From which equations A is found $= -\frac{14}{197}$; and B

$$\left(= -\frac{A}{14} \right) = \frac{1}{197} :$$

and, by substituting these values above,

$$\text{we have } -1 + 14 + \frac{14}{197} \times z + \frac{z^4}{197} = 0.$$

Which, by rejecting the exceeding small quantity $\frac{z^4}{197}$,

$$\text{becomes } -1 + 14 + \frac{14}{197} \times z = 0.$$

Hence $-197+2772z=0$;

and $z = \frac{197}{2772} = .0710678$, nearly: which value is true to the last decimal place: and, if more terms of the series $1+Az+Bz^2+Cz^3$ &c. had been taken, the conclusion would have been still exacter in proportion.

Ex. 2. Suppose the given equation (when prepared for a solution) to be $-2+5z-z^3=0$.

Here, if four terms of the general series $1+Az+Bz^2+Cz^3$ &c. be taken, and multiplied by $-2+5z-z^3$, our equation will be changed into the following one,

viz.

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$$\text{viz. } \left\{ \begin{array}{ccccccc} -2 & -2Az & -2Bz^2 & -2Cz^3 & * & & \\ * & +5z & +5Az^2 & +5Bz^3 & +5Cz^4 & & \\ * & * & * & -z^3 & -Az^4 & \& c. & \end{array} \right\} = 0.$$

Where, by comparing the homologous terms, we have $2B=5A$, $2C=5B-1$, and $5C=A$.

Hence, $25B-5 (=10C) = 2A$; but, by the first of these equations, $B = \frac{5A}{2}$; therefore $\frac{125A}{2} - 5 = 2A$, and consequently $A = \frac{10}{121}$.

Let this value be now wrote instead of A ; by which means our last equation for the value of z (neglecting the terms $-Bz^5 - Cz^6$) is reduced to $-2 + 5 - \frac{20}{121} \times z = 0$.

Whence z comes out $= \frac{242}{585} = 0.414$ nearly.

Let there be now given the general equation.
 $-p + az + bz^2 + cz^3 + dz^4 \& c. = 0$.

Then, multiplying by $(1 + Az)$ the two first terms of the series, only,

we get $\left\{ \begin{array}{l} -p + az + bz^2 \& c. \\ * -pAz + aAz^2 \& c. \end{array} \right\} = 0$.

Here, by making $b + aA = 0$, A is found $= -\frac{b}{a}$;

and our equation becomes $-p + a + \frac{bp}{a} \times z = 0$:

whence z is given $= \frac{p}{a + \frac{bp}{a}} = \frac{ap}{aa + bp}$.

The value of z , here determined, taking in two terms of the given series, $az + bz^2 + cz^3 \& c$. I call an approximation of the second degree (as the common method of converging series, which takes in the first

F f term

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term only, may, by the same rule, be called an approximation of the first degree.)

But to obtain an approximation of the third degree, or such an one as shall include three terms of the original series, let the given equation,

$-p+az+bz^2+cz^3 \mathcal{E}c. = 0$; be now multiplied by three terms of the assumed series $1+Ax+Bz^2 \mathcal{E}c$: whence there arises this

$$\text{new equation } \left\{ \begin{array}{l} -p+az+bz^2+cz^3 \mathcal{E}c. \\ * -Apz+Aaz^2+Abz^3 \mathcal{E}c. \\ * * -Bpz^2+Baz^3 \mathcal{E}c. \end{array} \right\} = 0.$$

Where, by equating the homologous terms, we have $b+Aa-Bp=0$, and $c+Ab+Ba=0$.

Let the former of these equations be multiplied by a , and the latter by p ; and then add the two products together; so shall

$ab+Aa^2+pc+Abp=0$; and consequently $A = -\frac{ab+cp}{aa+bp}$. Whence $z (= \frac{p}{a-pA})$ is likewise given.

To have an approximation of the fourth degree, four terms of the series $1+Az+Bz^2 \mathcal{E}c$. must be taken, for a multiplicator: by which means the equation given will be

$$\text{trans-} \left\{ \begin{array}{l} -p+az+bz^2+cz^3+dz^4 \mathcal{E}c. \\ * -Apz+Aaz^2+Abz^3+Acz^4 \mathcal{E}c. \\ * * -Bpz^2+Baz^3+Bbz^4 \mathcal{E}c. \\ * * * -Cpz^3+Caz^4 \mathcal{E}c. \end{array} \right\} = 0.$$

Here we have

$$B = \frac{Aa}{p} + \frac{b}{p},$$

$$C = \frac{Ba}{p} + \frac{Ab}{p} + \frac{c}{p},$$

$$0 = Ca + Bb + Ac + d,$$

from whence, exterminating C , there arises

$$\frac{Ba}{p} + \frac{Ab}{p} + \frac{c}{p} + \frac{Bb}{a} + \frac{Ac}{a} + \frac{d}{a} = 0,$$

or

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$$\text{or } \frac{a}{p} + \frac{b}{a} \times B + \frac{b}{p} + \frac{c}{a} \times A + \frac{c}{p} + \frac{d}{a} = 0.$$

Which, by substituting the value of B,

$$\text{becomes } \frac{a}{p} + \frac{b}{a} \times \frac{Aa}{p} + \frac{b}{p} + \frac{b}{p} + \frac{c}{a} \times A + \frac{c}{p} + \frac{d}{a} = 0;$$

$$\text{that is, } \frac{aa}{pp} + \frac{2b}{p} + \frac{c}{a} \times A + \frac{ab}{pp} + \frac{bb}{ap} + \frac{c}{p} + \frac{d}{a} = 0.$$

Multiply the whole by ap^2 ; then will $a^3 + 2abp + cpp$
 $\times A + a^2b + b^2p + acp + dp^2 = 0$;

$$\text{whence } A \text{ is found} = - \frac{aab + ac + bb \times p + dpp}{a^3 + 2bp + cpp};$$

and $z = \frac{p}{a - pA}$, as in the preceding case.

By the same method, an approximation of any higher degree, to take in as many terms of the proposed series as you please, may be derived.

For, it is evident, from above, that in all cases whatever,

$$B \text{ will be } = \frac{Aa}{p} + \frac{b}{p},$$

$$C = \frac{Ba}{p} + \frac{Ab}{p} + \frac{c}{p},$$

$$D = \frac{Ca}{p} + \frac{Bb}{p} + \frac{Ac}{p} + \frac{d}{p},$$

$$E = \frac{Da}{p} + \frac{Cb}{p} + \frac{Bc}{p} + \frac{Ad}{p} + \frac{e}{p},$$

$\mathfrak{E}c.$

$\mathfrak{E}c.$

where the last value is, always, equal to nothing.

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$$\text{Therefore, by making } \left\{ \begin{array}{l} Q = \frac{a}{p} \\ R = \frac{aQ+b}{p} \\ S = \frac{aR+bQ+c}{p} \\ T = \frac{aS+bR+cQ+d}{p} \\ V = \frac{aT+bS+cR+dQ+e}{p} \\ \text{\&c.} \end{array} \right.$$

$$\text{and } \left\{ \begin{array}{l} q = \frac{b}{p} \\ r = \frac{aq+c}{p} \\ s = \frac{ar+bq+d}{p} \\ t = \frac{as+br+cq+e}{p} \\ v = \frac{at+bs+cr+dq+f}{p} \\ \text{\&c.} \end{array} \right.$$

the value of A will be determined, by dividing the last of these quantities $q, r, s, t, \text{\&c.}$ by the corresponding quantity of the upper series $Q, R, S, \text{\&c.}$ and changing the sign of the quotient.

Whence we get $\frac{p}{a+p \times \frac{q}{Q}}, \frac{p}{a+p \times \frac{r}{R}}, \frac{p}{a+p \times \frac{s}{S}}, \text{\&c.}$

or, $\frac{1}{\frac{a}{p} + \frac{q}{Q}}, \frac{1}{\frac{a}{p} + \frac{r}{R}}, \frac{1}{\frac{a}{p} + \frac{s}{S}}, \text{\&c.}$ for so many different

values of z ; whereof each is more exact than the preceding one.

These

The Resolution of EQUATIONS. 221

These equations are easily derived from those above, expressing the relation of the quantities A, B, C, D, &c.

For, by writing Q and q, instead of their equals, in the first of those equations ($B = \frac{Aa}{p} + \frac{b}{p}$) it becomes $B = QA + q$.

Which value of B, wrote in the 2^d equation,

$$C = \frac{Ba}{p} + \frac{Ab}{p} + \frac{c}{p}, \text{ gives } C = \frac{aQA}{p} + \frac{aq}{p} + \frac{Ab}{p} + \frac{c}{p} = RA + r; \text{ by substituting R}$$

and r in the room of their equals.

Moreover the third equation, by substituting for C and B, becomes $D = \frac{aRA}{p} + \frac{ra}{p} + \frac{bQA}{p} + \frac{bq}{p} + \frac{Ac}{p} + \frac{d}{p} = SA + s$.

After the very same manner $E = TA + t$, $F = VA + v$, &c. And, by putting all these several values, successively, equal to nothing, we have $-\frac{q}{Q}$, $-\frac{r}{R}$, $-\frac{s}{S}$, &c. for so many different approximations of the value A. Which being substituted in the general equation $z = \frac{p}{a - pA}$, the very expressions before given, for the value of z, are obtained.

It will be proper, now, to shew the use of the several approximations, or theorems, derived above, by a few examples.

In

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In the first place, then, let the equation given be
 $x^3 = 10.$

This equation, by making $2+z=x$, will be transformed to $-2+12z+6z^2+z^3=0$: which being compared with the general equation

$-p+az+bz^2+cz^3+dz^4 \&c. = 0,$
 we have $p=2, a=12, b=6, c=1, d=0, \&c.$

Therefore (by the first of the three approximations, at p. 217) $-A (= \frac{b}{a})$ is found $= \frac{1}{2}.$

Whence $z (= \frac{p}{a-Ap})$ comes out $= \frac{2}{13} = 0.1538,$ nearly.

But, according to the second approximation, or theorem, the value of $-A (= \frac{ab+cp}{aa+bp})$ will be $=$
 $\frac{72+2}{144+12} = \frac{37}{78}$: and consequently that of $z (=$
 $\frac{p}{a-Ap}) = \frac{78}{505} = 0.15445.$

Lastly, the value of $-A$, according to the third approximation being $= \frac{a^2b + ac + bb \times p + dp^2}{a^3 + 2abp + cpp} =$
 $\frac{144 \times 6 + 96}{144 \times 12 + 144 \times 2 + 4} = \frac{36 \times 6 + 24}{36 \times 14 + 1} = \frac{48}{101},$ the
 corresponding value of z will therefore come out $=$
 $\frac{101}{654} = 0.154434$: which number is true in all its places.

The very same conclusions will likewise be brought out from the general solution.

For,

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For, p being here $= 2$, $a = 12$, $b = 6$, $c = 1$,
 $d = 0$, &c. (as before.)

we have $Q \left(= \frac{a}{p} \right) = 6$,

$$R \left(\frac{aQ + b}{p} \right) = \frac{12 \times 6 + 6}{2} = 39,$$

$$S \left(\frac{aR + bQ + c}{p} \right) = \frac{505}{2},$$

&c.

Also, $q \left(\frac{b}{p} \right) = 3$,

$$r \left(\frac{aq + c}{p} \right) = \frac{37}{2},$$

$$s \left(\frac{ar + bq + d}{p} \right) = 120,$$

&c.

Therefore $\frac{q}{Q} = \frac{1}{2}$, $\frac{r}{R} = \frac{37}{78}$, $\frac{s}{S} = \frac{240}{505} = \frac{48}{101}$; and con-

sequently $\frac{1}{\frac{a}{p} + \frac{q}{Q}} = \frac{1}{6 + \frac{1}{2}} = \frac{2}{13}$, &c. as before.

For a second example, suppose $10z - z^3 = 2$, or
 $-2 + 10z - z^3 = 0$.

In this case $p = 2$, $a = 10$, $b = 0$, $c = -1$, $d = 0$, &c.
 And therefore, for an approximation of the fourth

degree, we have $-A \left(\frac{aab + ac + bb \times p + dpp}{a^3 + 2abp + cpp} \right) =$
 $\frac{-20}{1000 - 4} = \frac{-5}{249}$; and consequently $z \left(= \frac{p}{a - pA} \right) =$
 $\frac{249}{1240} = 0.2008045$; which value is true to the last
 figure.

For

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For a third example, let there be given the equation
 $x^3 + 36x^2 + 432x = 2272$.

Here the value of x appears to be about 4; let therefore $4+z$ be wrote for x ; whence our equation is reduced to

$$96 + 768z + 48z^2 + z^3 = 0.$$

Which being compared with

$$-p + az + bz^2 + cz^3 + dz^4 \text{ &c.} = 0,$$

we have, in this case,

$$p = -96, a = 768, b = 48, c = 1, d = 0, \text{ &c.}$$

$$\text{or, } \frac{a}{p} = -8, \frac{b}{p} = -\frac{1}{2}, \frac{c}{p} = -\frac{1}{96}, \text{ &c.}$$

$$\text{Hence } Q \left(\frac{a}{p} \right) = -8,$$

$$R \left(\frac{aQ}{p} + \frac{b}{p} \right) = 64 - \frac{1}{2} = \frac{127}{2},$$

$$S \left(\frac{aR}{p} + \frac{bQ}{p} + \frac{c}{p} \right) = -\frac{48385}{96};$$

$$\text{Also } q \left(\frac{b}{p} \right) = -\frac{1}{2},$$

$$r \left(\frac{aq}{p} + \frac{c}{p} \right) = 4 - \frac{1}{96} = \frac{383}{96},$$

$$s \left(\frac{ar}{p} + \frac{bq}{p} + \frac{d}{p} \right) = -\frac{380}{12},$$

$$\text{Therefore } \frac{q}{Q} = \frac{1}{16}, \frac{r}{R} = \frac{383}{127 \times 48} = \frac{383}{6096}, \frac{s}{S} = \frac{380 \times 8}{48385} = \frac{3040}{48385};$$

$$\text{From whence } z = -\frac{16}{127} = -0.12598, \text{ nearly;}$$

$$\text{or, } z = -\frac{6096}{48385} = -0.1259894, \text{ more nearly,}$$

$$\text{or, } z = -\frac{9677}{76808} = -0.1259894802, \text{ still nearer.}$$

Lastly,

The RESOLUTION of EQUATIONS 225

Lastly, let there be given the equation

$$\frac{x^2}{2} - \frac{x^4}{2 \cdot 3 \cdot 4} + \frac{x^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} - \frac{x^8}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} \mathcal{C}c. = \frac{1}{2}.$$

Then, making $z = x^2$, $\mathcal{C}c.$ we have

$$-1 + z - \frac{z^2}{12} + \frac{z^3}{12 \times 30} - \frac{z^4}{12 \times 30 \times 56} \mathcal{C}c. = 0.$$

Here $p = 1$, $a = 1$, $b = -\frac{1}{12}$, $c = \frac{1}{12 \times 30}$, $d = -\frac{1}{12 \times 30 \times 56}$: $\mathcal{C}c.$

And, by substituting these values in the third general approximation (*vid. p. 219*), we have $-A$

$$\left(= \frac{aab + ac + bb \times p + dpp}{aaa + 2abp + cpp} \right) \\ = \frac{-\frac{1}{12} + \frac{1}{12 \times 30} + \frac{1}{12 \times 12} - \frac{1}{12 \times 30 \times 56}}{1 - \frac{2}{12} + \frac{1}{12 \times 30}} = - \\ \frac{30 \times 56 - 56 - 140 + 1}{10 \times 30 \times 56 + 56} = -\frac{1485}{16856}.$$

Therefore $z \left(= \frac{p}{a - pA} \right) = \frac{16856}{15371} = 1.09661$;

and $x (\sqrt{z}) = 1.04719$.

After the same manner the roots of other equations may be approximated: but I shall here shew, how the general theorems themselves may be rendered more commodious, for certain particular cases, by means of a proper transformation.

It is known, if two quantities be, respectively, increased, or decreased by two other, small, quantities, nearly in the same ratio with the two first, that the sums, or differences will still be in the same ratio with the two first quantities, very near.

G g

Wherefore

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Wherefore, seeing the numerator of the fraction $\frac{ab+cp}{aa+bp}$, expressing the 2^d. general value of $-A$, (*vid. p. 218*) is in proportion to the denominator, nearly, as b to a , or cp to $\frac{acp}{b}$; let cp be therefore taken from the numerator, and $\frac{acp}{b}$ from the denominator; agreeable to the above observation: by which means the fraction itself will be transformed to $\frac{ab}{aa+bp-\frac{acp}{b}} =$

$$\frac{\frac{b}{\frac{b}{a}-\frac{c}{b}}}{a+\frac{b}{a}-\frac{c}{b} \times p} = \frac{b}{a+rp}; \text{ supposing } r = \frac{b}{a} - \frac{c}{b}.$$

And, in the very same manner the fraction $\frac{aab+ac+bb \times p+dp}{aaa+2abp+cpp}$, expressing the third value of $-A$, is changed to $\frac{aab+ac+bb \times p}{aaa+2abp+cpp-\frac{adpp}{b}} =$

$$\frac{aab+ac+bb \times p}{a^3+2abp+sp^2} \text{ (by putting } s=c-\frac{ad}{b}\text{)}.$$

But this last value is still capable of a further reduction: for, the ratio of the numerator to the denominator being that of $\frac{bsp}{r}$ to $\frac{asp}{r}+sp^2$, nearly (as appears from the preceding case) let, therefore, the former of these quantities be subtracted from the numerator, and the latter from the denominator: whence the fraction itself becomes

$$\frac{a^2b+ac+bb \times p - \frac{bsp}{r}}{a^3+2abp-\frac{asp}{r}} = \frac{ab+c+\frac{bb}{a}-\frac{bs}{ra} \times p}{a^2+2b-\frac{s}{r} \times p} =$$

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$$= \frac{ab + c + bw \times p}{aa + b + aw \times p}; \text{ by putting } w = \frac{b}{a} - \frac{s}{ar} (= \frac{b}{a} - \frac{cb - ad}{bb - ac}).$$

Now, to exemplify the use of the theorems thus transformed, let the equation $96 + 768z + 48z^2 + z^3 = 0$ (given at p. 224) be here resumed :

Then, in this case, p being $= -96$, $a = 768$, $b = 48$, $c = 1$, $d = 0$, &c. we have $r (= \frac{b}{a} - \frac{c}{b}) = \frac{1}{24}$, $s (= c - \frac{ad}{b}) = 1$, and $w (= \frac{b}{a} - \frac{s}{ar}) = \frac{1}{32}$.

Whence, according to the first approximation,

$$-A (= \frac{b}{a + rp}) = \frac{48}{768 - 4} = \frac{12}{191}.$$

And, according to the second,

$$-A (= \frac{ab + c + bw \times p}{aa + b + aw \times p}) = \frac{768 \times 48 + 1 + \frac{1}{2} \times -96}{768 \times 768 + 48 + 24 \times -96} = \frac{768 - 5}{768 \times 16 - 72 \times 2} \text{ (by dividing every term by 48)} = \frac{763}{12144}.$$

Hence the value of $z (= \frac{1}{\frac{a}{p} - A})$ comes out $= -$

$$\frac{191}{1516} = -0.1259894, \text{ nearly; or equal to } -\frac{12144}{96389} = -0.1259894802, \text{ more nearly.}$$

These conclusions agree with those before given, by the former method. But the last general approximations, containing the fewest dimensions of the quantity p , will commonly be found to have the advantage, in point of expedition, when the value of that quantity consists of several decimal

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places, and also when the coefficients a, b, c, d , of the powers of the unknown quantity z , are related to each other according to some unknown law.

Of this kind are the coefficients of such series as arise in extracting the roots of pure powers; and in these cases the general theorems, or equations themselves, are capable of being rendered still more commodious.

Let there be assumed the equation $x^n = k$ (which includes all the cases of pure, or simple powers, according to the value of the exponent n).

Then, by assuming r nearly equal to x , and making $r \times \overline{1+z} = x$, we shall have $r^n \times \overline{1+z}^n = k$:

and therefore $-\frac{k}{r^n} + \overline{1+z}^n = 0$; that is

$$-\frac{k}{r^n} + 1 + nz + \frac{n}{1} \times \frac{n-1}{2} z^2 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{2} z^3$$

$$\mathcal{G}c. = 0; \text{ or lastly } -p + z + \frac{n-1}{2} z^2 + \frac{n-1}{2} \times \frac{n-2}{3}$$

$$z^3 + \mathcal{G}c. = 0; \text{ by dividing the whole by } n, \text{ and putting}$$

$$p = \frac{k - r^n}{nr^n}.$$

Here (by a comparison with the general equation $-p + az + bz^2 \mathcal{G}c. = 0$) we have $a = 1$, $b = \frac{n-1}{2}$, $c = \frac{n-1}{2} \times \frac{n-2}{3}$, $d = \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, $\mathcal{G}c.$

$$\text{Therf. } r \left(\frac{b}{a} - \frac{c}{b} \right) = \frac{n-1}{2} - \frac{n-2}{3} = \frac{n+1}{6},$$

$$s \left(c - \frac{ad}{b} \right) = \frac{n-2}{3} \times \frac{n-1}{2} - \frac{n-3}{4} = \frac{n-2}{3} \times \frac{n+1}{4},$$

$$w \left(\frac{b}{a} - \frac{s}{ar} \right) = \frac{n-1}{2} - \frac{n-2}{2} = \frac{1}{2},$$

Whence,

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Whence, for an approximation of the third degree,

$$-A \left(= \frac{b}{a+rp} \right) = \frac{\frac{1}{2} \times n - 1}{1 + n + 1 \times \frac{1}{6}p};$$

$$\text{and } z \left(= \frac{p}{a-Ap} \right) = \frac{p + n + 1 \times \frac{1}{6}p^2}{1 + 2n - 1 \times \frac{1}{3}p}.$$

But, for an approximation of the fourth degree,

$$-A \left(= \frac{ab+c+bw \times p}{aa+b+aw \times p} \right) = \frac{\frac{1}{2} \times n - 1 + n - 1 \times 2n - 1 \times \frac{1}{12}p}{1 + \frac{1}{2}np};$$

$$\text{and } z = \frac{p + \frac{1}{2}npp}{1 + \frac{2n-1}{2} \times p + \frac{2n-1 \times n-1}{12} \times pp}.$$

Hence it is manifest that the root x , of the proposed equation $x^n=k$, is equal to

$$r + \frac{rp \times 1 + n - 1 \times \frac{1}{6}p}{1 + 2n - 1 \times \frac{1}{3}p}, \text{ nearly};$$

$$\text{or, equal to, } r + \frac{rp \times 1 + \frac{1}{2}np}{1 + \frac{2n-1}{2} \times p + \frac{2n-1 \times n-1}{12} \times p^2},$$

more nearly.

But both these expressions, in cases where p is a proper fraction, will be better adapted to practice by making $\frac{nr^n}{k-r^n}=v \left(= \frac{1}{p} \right)$, and substituting $\frac{1}{v}$ for p , its equal: whence (after proper reduction)

$$x = r + \frac{r}{v} \times \frac{6v+n+1}{6v+4n-2}, \text{ nearly};$$

$$\text{or, } x = r + \frac{r \times 2v + n}{v \times 2v + 2n - 1 + \frac{1}{6} \times 2n - 1 \times n - 1}, \text{ more nearly.}$$

To

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To shew now the use and great exactness of these last approximations, by an example, let it be proposed to extract the square root of 441. Here the general equation $x^n = k$, becoming $x^2 = 441$, we have $n = 2$, $k = 441$, and $v \left(= \frac{nr^n}{k-r^n} \right) = \frac{800}{41}$; supposing r to be assumed $= 20$.

Therefore, by the first approximation,

$$x = 20 + \frac{41}{40} \times \frac{1641}{1682} = 20 + \frac{67281}{67280} = 21 + \frac{1}{67280},$$
 nearly.

And, by the second, $x = 20 + \frac{2758480}{2758481} = 21 - \frac{1}{2758481}$, more nearly.

Again, let there be given the equation $x^3 = 500$: then, assuming $r = 8$, we have $v \left(\frac{nr^n}{k-r^n} \right) = \frac{3 \times 512}{-12} = -128$.

Hence, by the first approximation,

$$x = 8 - \frac{1}{16} \times \frac{-768+4}{-768+10} = 8 - \frac{191}{3032} = 7.93700527,$$
 nearly.

And, by the second, $x = 8 - \frac{8 \times 253}{128 \times 251 + \frac{1}{2}} = 8 - \frac{6072}{9638} = 7.9370052599$, more nearly.

All the different approximations hitherto delivered were, originally, derived by multiplying the given equation into a certain number of terms of the assumed series $1 + Az + Bz^2 + Cz^3$ &c. But there are other methods (though, perhaps, none so general) by which the second, third, &c. dimensions of the unknown quantity may, in like sort, be destroyed (without assuming

The Resolution of EQUATIONS. 231

fuming any series) and from thence the value of that quantity approximated, to what degree of exactness you please.

Let there, for instance, be given the equation,
 $2z + z^2 = 1$, or $z^2 = 1 - 2z$.

Then, by squaring both sides thereof, there arises
 $z^4 = 1 - 4z + 4z^2$.

And if, instead of the last term $4z^2$, its equal,
 $4 \times 1 - 2z$, be substituted, you will have $z^4 = 5 - 12z$:
 this, squared, gives $z^8 = 25 - 120z + 144z^2 = 169 -$
 $408z$, by substituting for z^2 , as before.

Here, rejecting z^8 (on account of its smallness in
 comparison of the other terms) we have $408z = 169$,
 and therefore $z = \frac{169}{408} = 0,414213$, nearly; which
 is true to the last figure, inclusive. But, if you would
 have the answer still nearer the truth, let the above
 equation $z^8 = 169 - 408z$ be, either, multiplied,
 again, by itself, or into some one of the preceding
 ones. Thus, if it be multiplied into $z^4 = 5 - 12z$,
 you will have

$$z^{12} = 845 - 4068z + 4896z^2 = 5741 - 13860z:$$

where, z^{12} being rejected, z is found $= \frac{5741}{13860} =$
 0.41421356 , &c.

Again, if there be given the equation $z^3 = 3z - p$;
 then, by squaring both sides thereof, we have
 $z^6 = 9z^2 - 6pz + p^2$:
 and therefore $z^7 = 9z^3 - 6pz^2 + p^2z = -6pz^2 +$
 $pp + 27 \times z - 9p$; by writing $27z - 9p$ instead of
 its equal $9z^3$.

Now, to the triple of this last equation, let the 2^d .
 equation, multiplied by $2p$, be added:

By

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By which means z^2 will be exterminated, and you then will have $3z^7 + 2pz^6 = 81 - 9pp \times z - 27p + 2p^2$.

Whence (rejecting $3z^7 + 2pz^6$) the value of z is found

$$= \frac{27 - 2pp \times p}{9 - pp \times 9}, \text{ nearly.}$$

Various other expedients might be used, to exterminate the 2^d , 3^d , &c. powers of the unknown quantity; but what is already delivered may suffice.

P A R T V.

GIVING

*Some Account of the Nature of FLUXIONS;
together with the Investigation of the funda-
mental Rules.*

1. **I**N the Doctrine of Fluxions all kinds of magnitudes are considered as generated by the continual motion of some of their bounds or extremes; as a line by the motion of a point; a surface by the motion of a line; and a solid by the motion of a surface. So likewise time may be considered as represented by a line, increasing uniformly by the motion of a point: and, as quantities of all kinds whatever are capable of increase and decrease, they may be represented, in like manner, by lines, surfaces, or solids, conceived to be generated by motion.

2. Every quantity thus generated is called a *fluent*, or *flowing quantity*: and the magnitude by which any flowing quantity would be uniformly increased, in a given time, with the generating celerity at any proposed position, or instant * (was it from thence to continue invariable) is the *fluxion* of the said quantity at that position, or instant.

3. Thus,

* Some Authors define the generating celerity itself (and not the magnitude it would produce) to be the fluxion; but use that magnitude as the measure of the said celerity or fluxion; which is, in effect, coming to the same thing.

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3. Thus, let a point m be conceived to move from A , and thereby generate a right-line Am , with a motion any how regulated; and suppose the celerity

thereof, at any proposed position R , to be

such, as would, (was it to continue invariable from that position) be sufficient to describe, or pass uniformly over the distance Rr , in the given time allowed for the fluxion; then will the said distance Rr truly express the required fluxion of the flowing line Am , in that position.

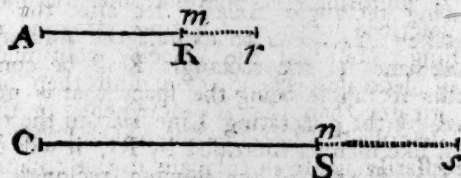
4. It appears from hence, that, when the generating motion is uniform, the fluxion, and the increment actually described in the given time, are one and the same thing: but, if the velocity continually increases or decreases, the fluxion must then be either less, or greater than the said increment, or the space actually described: since an increase of the velocity must necessarily cause an increase in the distance gone over, and *vice versa*.

5. If

Although, in forming a just and distinct conception of the nature, and quantity of a fluxion, the consideration of time is, absolutely, necessary (on which, even, our ideas of velocity depend), yet in the business and application of fluxions, it is not always requisite, that some, vulgar (or common) measure of time (as a second, minute, hour, &c.) should be propounded for the production of the fluxions of the quantities under consideration. A line generated by the uniform motion of a point, it is observed above, may be taken as a proper representative, or measure of time: and that interval of time (be it what it will) wherein the line, so generated, is augmented by any length, or fluxion, assigned, may be taken as the time understood in the definition, allowed for the production of the fluxions of all other quantities that have any relation to, or dependence upon, the said uniformly generated line. And these fluxions themselves, by means of the said relation and the given length, or fluxion, may be also truly exhibited, independent of any particular, known measure of time; as will hereafter, be fully made to appear.

5. It appears moreover, from the above definition, that quantities, which flow, or increase together, so as to continue, *always*, in a constant ratio, have their fluxions, likewise, in *the same* constant ratio.

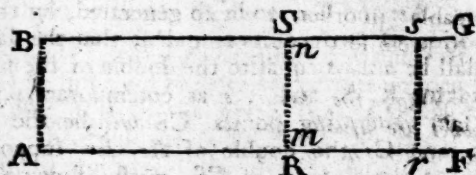
To illustrate this by a particular example, suppose two lines, Am and Cn , to be so generated, by the uniform motion of two points m and n , that the latter of them shall be *always* equal to the double of the former: then, taking R , S , and r , s as cotemporary positions of the said generating points, CS will be the double of AR , and Cs the double of Ar , by supposition; whence Ss , the fluxion of CS , must of consequence be the double of Rr , the fluxion of AR .



It is equally plain, on the other hand, that if the ratio of the fluents Am , Cn is variable, that of the fluxions must also vary. Thus, if, while the point m continues to move uniformly on, at the rate of one inch (foot, yard, &c.) in a second of time, the motion of the other Point n be so regulated that the number of inches (feet, yards, &c.) in the flowing line Cn generated thereby, may be *always* equal to the square of the number of those in Am described by the former point m ; then, in this case, it is manifest, that the ratio of the fluents Am , Cn is a variable one; and that the ratio of the fluxions varies also; seeing the distances 1, 4, 9, 16, 25, &c. described in 1, 2, 3, 4, 5, &c. seconds of time, by the point n , increase much faster in proportion than 1, 2, 3, 4, 5, &c. the corresponding distances gone over by the other point m , moving uniformly.

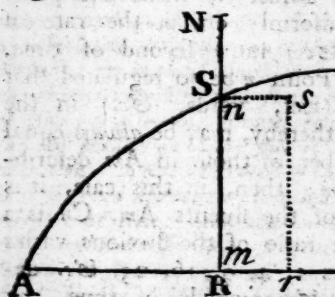
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6. Hitherto regard has been had to the fluxions of lines: but the fluxions of superficies and solids are considered in the same manner, and are comprehended with equal facility. Let a given right-line mn



be conceived to move parallel to itself, with an equable motion, from the position AB , and thereby generate the flowing rectangle $AmnB$; let also the distance Rr be taken (*as above*) to express the fluxion of the base Am , and let the rectangle $Rmns$ be completed: then, this rectangle being the space that is uniformly described by the generating Line mn , in the time that Am is in like manner increased by Rr , it will therefore be the true fluxion of the flowing rectangle Am , by the definition. *Art. 2.*

7. The generation, and the fluxion of any triangular, or curvilinear, space ASR , are conceived in



like manner; by supposing a right-line mN to be carried along, continually parallel to itself, so that the intercepted part thereof mn (which is itself a variable quantity) may pass over, and thereby generate, the space ARS propounded. And the fluxion of the space thus generated, if Rr be taken as the fluxion of the base (or abscissa) Am , will be truly expressed by the rectangle (Rr) under Rr and RS ; as we shall have occasion to shew more at large hereafter.

8. From

8. From what has been thus far delivered, it will not be difficult to form a just idea of the fluxion of a solid: but it is time we now come to shew the manner of determining the fluxions of algebraic quantities; by means whereof all others, of what kind soever, are explicable: in order to which it will be requisite, first of all, to premise the following observations.

1. That, the final letters u, w, x, y, z , of the alphabet are usually put for variable quantities; and the initial letters a, b, c, d , &c. for invariable ones: thus, the variable base Am of the flowing rectangle $AmnB$ (in Art. 6.) may be represented by x , and the invariable altitude mn , by a .

2. That, the fluxion of a quantity represented by a single letter is commonly expressed by the same letter with a dot, or full-point, over it: thus the fluxion of x is denoted by $\dot{x}$; and the fluxion of y by $\dot{y}$.

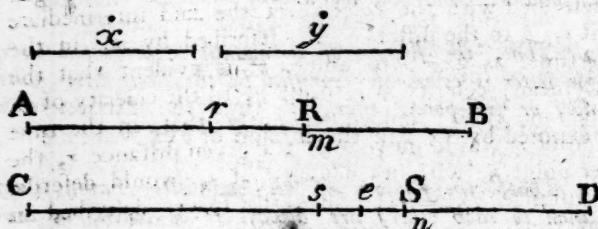
3. That, the fluxions of all quantities (having any relation to each other) are always to be taken, as contemporaneous, or such as may be generated together, with their respective celerities, in one and the same time.

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PROPOSITION I.

9. *The Fluxion of any variable Algebraic Quantity being given, it is proposed to find the Fluxion of the Square of that Quantity.*

Let two points m and n be supposed to move, at the same time, from two other, fixed, points A and C , along the right-lines AB and CD , in such a manner, that the measure of the distance Cn , described by the latter, may be, *always*, equal to the square of the cotemporary distance Am described by the former point m , moving with an equable celerity.



Moreover, let R and S be any two cotemporary positions of the said points; and, supposing the described distances AR and CS to be denoted by x and y , let the lines x and y be taken to represent the spaces that *would be* uniformly passed over, in the same given time, with the celerities of the said points at R and S : so shall those lines express the fluxions of the variable quantities Am and Cn , when the generating points m and n arrive at the foresaid cotemporary positions R and S (by the definition, Art. 2).

Furthermore, if r and s be considered as any other corresponding places of the proposed points, and the interval rR be denoted by v ; then, AR being $=x$, and $Ar = x - v$, we shall have $CS (=y) = x^2$, and $Cs = x^2 - v^2$, by hypothesis; and consequently $Ss (=CS - Cs) = 2xv - vv$.

From

From which it appears, that, while the former point m moves, uniformly, over the distance v , the other point n passeth over a space expressed by $2xv - vv$.

But this last distance, since the velocity of the generating point n increases continually (*see Art. 5.*) is less than the space that would be uniformly described, in the same time, with the velocity at S ; and greater than that which would be described with the velocity at s ; and, therefore, is equal to, and may be taken to express, the space which *might be* uniformly gone over by the celerity at some intermediate point e , between s and S , in the same time.

Therefore, seeing the distance ($2xv - vv$) that might be described with the celerity at the said intermediate point e , is to the distance (v) described by m , in the same time, as $2x - v$ to unity, it is evident that the said (mean) celerity at e , must be to the celerity of m , in the same ratio; and consequently, that, in the time the point m would move over the given distance s , the other point n , with its velocity at e , would describe the distance $2xs - vs$: since the spaces described in equal times, by uniform motions are known to be as the velocities of the said motions,

This

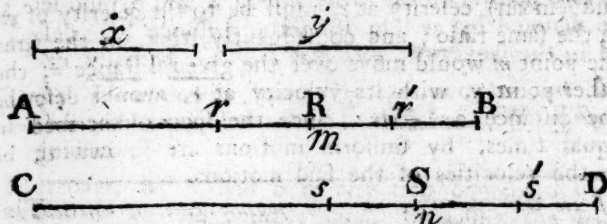
The above method of investigation hath been represented, as bearing a near affinity to the method of Prime and Ultimate Ratios: against which so many objections have been started, by the celebrated author of the Analyst, as to embarrass and stagger a great number of persons; who are not well apprized how far the said objections are justifiable, nor wherein their main force consists. It is not my design to take a part in a dispute, on which enough hath been already said by others: though, that the method itself is perfectly scientific, I believe no one, that understands it, will deny; but whether the great inventor has therein expressed himself with all the caution and accuracy he was capable of, is another question. Had he called that a LIMITING-RATIO which he names an Ultimate-one, the ingenious author above mentioned might not, perhaps, have found room for his Ghosts of departed Quantities. However, be this as it will,
those

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This being determined, let r be now supposed to coincide with R , and s with S , by means of the arrival of the generating points at R and S ; then e , being always between s and S , will likewise coincide with S ; and the forefaid distance $2xs - vx$, that might be uniformly described with the velocity at e (now at S) will become, barely, $2xs$; which (by *Art. 2*) is equal to (y) the fluxion of Cx or x^2 . From whence it appears, that the fluxion of the square of any variable, or flowing quantity is found by multiplying twice the root, or quantity itself, into its fluxion.

The same otherwise.

10. Let the distance Ar be denoted by u , and let other things remain as before: then, Cs being =



xx , and $Cs = uu$, by supposition, the distance sS , described in the same time with rR ($= x - u$) will therefore be truly expressed by $xx - uu$, or its equal $x + u$ $\times x - u$. Which distance, as the velocity of n continually increases, must be greater than that which would be uniformly described, with the celerity at s , in the same time. Whence it is evident, that the velocity

these objections have nothing at all to do with our Method of investigation given above. The pressing (and only) difficulty, in the business of ultimate ratios, consists, it is known, in considering the values of the quantities compared, in their ultimate state, wherein their ratio is supposed to be taken: for

velocity at s is to the, uniform, velocity of m , in a less ratio than that of $x+u \times x-u$ to $x-u$, or of $x+u$ to unity, or lastly, of $x+u \times x$ to x (because the velocities of uniform motions are as the spaces described in equal times.) Therefore, since the uniform velocity of m (measured by the distance moved over, in a given time) is defined by x it is plain that the measure of the velocity of the other point at s (or the distance that *might be* uniformly described in the same given time) will be less than $x+u \times x$. Which quantity being, itself, less than $2x \times x$ (because u is less than x) the celerity at s must consequently be less than $2xx$, take the point s where you will on this side of S .

Let now r' and s' be any other cotemporary positions of the two points, on the other side of R and S , and let Ar' be denoted by w : so shall the distance Ss' , described in the same time with Rr' ($w-x$), be truly expressed by $w-w-x$, or its equal $w+x \times w-x$ (by hypothesis.) Which distance, as the velocity of the describing

for, if we look upon them as real magnitude, it is objected, that their ratio will not strictly agree with the ultimate ratio assigned: and if, on the other hand, they be taken as mere nothings, we then lose the very idea of proportion. But our investigation, as is already observed, is not embarrassed with any such difficulty: for, though the distance Ss grows, indeed, less and less, continually, and even vanishes when the generating point n arrives at S ; yet the velocity of that point, which is the quantity in question, neither vanishes, nor assumes a new law; but still continues to increase in the same manner as before.—It may, possibly, be objected, that, as the measure of the said velocity is, originally, derived by means of the distance nS , we cannot retain a clear idea of it, when that distance is vanished out of the equation. But, with equal reason, it might be urged, that, we can have no just conception of the dimensions and true proportion of a building, after the scaffolding by means of which it was raised, is taken away.

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cribing point n continually increases, must, evidently, be less than that which would uniformly arise from the celerity at s , in the same time. Whence it is also plain that the said velocity at s , is to the uniform velocity of m , in a greater ratio than that of $w+x \times w-x$ to $w-x$, or of $w+x \times x$ to x . But the quantity $w+x \times x$ is, itself, greater than $2x \times x$, because w is greater than x ; and so the measure of the said velocity at s must consequently be greater than $2xx$.

Therefore, since the velocity increases continually, from C to D ; and seeing the measure thereof, before the arrival of the generating point at S , is every where less, and afterwards every where greater, than $2xx$; it is manifest, that, at S , it can be neither lesser nor greater, but must have, or pass through, the very value, or degree, expressed by $2xx$. Q. E. I.

If the line Am (x) be supposed to be generated with an accelerated, or a retarded motion, instead of an uniform one, it will readily appear, from the first of the foregoing methods, that the required fluxion of x^2 , supposing x to denote the measure of the velocity at R , will, *still*, be expounded by $2xx$.

For the spaces rR (v) and sS ($2xv-vv$) described in the same time, being to each other, in the ratio of x to $2xx-vx$, the mean celerities of the generating motions, at certain intermediate points between the extreme ones r , R , and s , S , must be likewise, in that ratio: which ratio, when v becomes $=0$, and the points coincide, will become that of x to $2xx$.

PROPOSITION

PROPOSITION II.

11. *The Fluxions, $\dot{x}$ and $\dot{y}$, of two flowing Quantities, x and y , being given; it is proposed to determine the Fluxion of the Rectangle, or Product, xy , of the said Quantities.*

Let z be, always, equal to the sum of the two proposed quantities x and y : then, the fluxions of equal quantities being also equal, $\dot{z}$ must be $=\dot{x}+\dot{y}$. Moreover, since z is $=x+y$, we shall have $zz = xx + 2xy + yy$; and therefore $\dot{z}z = \dot{x}x + \dot{z}x + \dot{z}y + \dot{y}y$; But the fluxion of $\dot{z}z - \dot{x}x - \dot{y}y$ (and consequently that of its equal xy) appears, from the preceding proposition, to be $\dot{z}z - \dot{x}x - \dot{y}y$: which, by writing $x+\dot{y}$, and $\dot{x}+\dot{y}$, in the room of their equals z and $\dot{z}$, will become $(x+\dot{y}) \times (\dot{x}+\dot{y}) - \dot{x}x - \dot{y}y = \dot{y}x + \dot{x}y$, the required fluxion of xy . Hence it is apparent that the fluxion of the product, or rectangle, of any two flowing quantities, is expressed by the sum of the products arising from the multiplication of each quantity into the fluxion of the other.

12. From the fluxion of a rectangle, above determined, the fluxion of a fraction, $\frac{z}{y}$, is very easily deduced.

For, by putting $z=\frac{z}{y}$ (the proposed fraction) and multiplying both sides of the equation by y we have $xy=z$; and therefore $\dot{x}y + y\dot{x} = \dot{z}$, as above, (the fluxions of equal quantities being, necessarily, equal.) From this equation, by transposing $\dot{x}y$, and dividing the whole by y , we get $\dot{x} = \frac{\dot{z}}{y} - \frac{x\dot{y}}{y}$: this, by writing

$\frac{z}{y}$ in the room of its equal x , becomes $\dot{x} = \frac{\dot{z}}{y} -$

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$\frac{x\dot{y}}{yy} = \frac{y\dot{z} - z\dot{y}}{yy}$: which value is therefore the true fluxion of x , or, its equal, $\frac{z}{y}$, the fraction proposed.

13. Moreover, from the fluxion of a rectangle, the fluxion of the continual product of three, four, five, or any other number, of flowing quantities, may be determined.

Thus, let the fluxion of yzu , where the number of factors is 3, be first required: then, by putting $x = zu$, our given expression will be reduced to yx ; and its fluxion will be $y\dot{x} + x\dot{y}$ (by Prop. 2.) But, x being $= zu$, and therefore $\dot{x} = z\dot{u} + u\dot{z}$ (by the same), if these values be substituted in $y\dot{x} + x\dot{y}$, it will become $y \times z\dot{u} + u\dot{z} + zu\dot{y} = yz\dot{u} + yu\dot{z} + zu\dot{y}$, the true fluxion of yzu , required.

Again, if the fluxion of $yzuw$, where the number of factors is four, was to be demanded; then, by making $x = zuw$, the quantity proposed will be reduced to yx ; and its fluxion will therefore be expressed by $y\dot{x} + x\dot{y}$; which, because x is $= zuw$, and $\dot{x} = z\dot{u}w + u\dot{z}w + zu\dot{w}$ (as appears from above) will be likewise expressed by $y \times z\dot{u}w + u\dot{z}w + zu\dot{w} + zuw\dot{y}$, or $yzw\dot{u} + yzu\dot{w} + yzw\dot{u} + yzu\dot{w}$.

In the same manner the fluxion of $yzvws$, will appear to be $yzvws\dot{u} + yzvws\dot{v} + yzvws\dot{w} + yzvws\dot{s}$: and so of others.

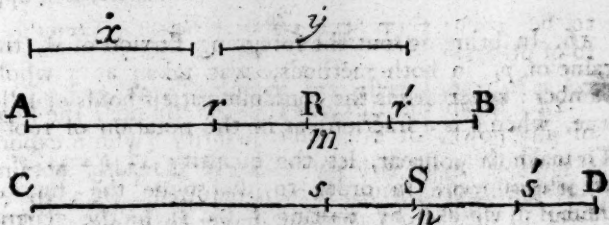
14. From the fluxions thus determined, the fluxion of any power of a variable quantity (whose exponent is a whole positive number) is very readily obtained; nothing more being herein required, than to consider all the factors as equal among themselves. Thus, the

the fluxion of yvw being found $yvw + yvw + yvw$, it is plain, if both v and w be supposed equal to y , that the fluent, or quantity propounded will become y^3 , and its fluxion $yy' + y'y + yyy = 3y^2y$.

Thus, also, because the fluxion of $yzvw$ is $yzvw + yzvw + yzvw$, it appears that the fluxion of y^4 will be truly expressed by $4y^3y$. And, from the given fluxion of $yzvw$, that of y^5 will in like manner appear to be $5y^4y$. From whence the law of continuation is manifest; the fluxion of y^p being universally expounded by $py^{p-1}y$.

15. This last general conclusion, which is of very great importance in the business of fluxions, being the result of several deductions, whereby its truth and evidence may, perhaps, lose a part of their force, a more direct investigation of the same may not here be amiss; though the former method will, doubtless, appear the most easy and proper for beginners, to whom, the manner of working by general indices, is not plain and familiar.

Conceive two points m and n to move, at the same time, from two other points A and C , along the right-lines AB and CD ; and let every thing be supposed as in Prop. 1; only, let the measure of the distance described by the point n be, always, equal to the p power, (instead of the square) of that described by the other point m , moving uniformly.



Then,

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Then, since by hypothesis, the value of CS is here $= x^p$, and that of Cs $= u^p$, (see the second solution to the foregoing prop.) it is evident that the distance SS, described in the same time with R, ($= x - u$), will be truly defined by $x^p - u^p$, or its equal $x - u$

$$\times x^{p-1} + x^{p-2}u + x^{p-3}u^2 + x^{p-4}u^3 + \dots + u^{p-1}.$$

Whence, supposing $\dot{x}$ to denote the measure of the uniform velocity of m , it will appear, by reasoning as in the said proposition, that the measure of the velocity of n , at any place s , on this side of S, must be less than $\dot{x} \times x^{p-1} + x^{p-2}u + x^{p-3}u^2 + \dots + u^{p-1}$, and consequently less than $\dot{x} \times px^{p-1}$; seeing each of the (p) terms of the said series (the first only excepted) is less than x^{p-1} , u being less than x .

Again, by considering r' and s' as two other, cotemporary positions, beyond R and S, and reasoning in the same manner, the measure of the velocity at s' will appear to be greater, now, than the abovesaid quantity $\dot{x} \times px^{p-1}$.

Therefore, as the velocity increases continually, and seeing the value thereof, before the point arrives at S, is every where less; and afterwards, every where greater, than $\dot{x} \times px^{p-1}$ (or $px^{p-1}\dot{x}$) it is evident, that, at S, it must be neither lesser nor greater, but exactly equal to $px^{p-1}\dot{x}$: which quantity is therefore the true fluxion of x^p : and agrees exactly with that determined above.

16. In bringing out the foregoing fluxion of x^p , the value of p , in both methods, was taken as a whole number: nevertheless the conclusion itself holds equally true, when p is a fraction, as in the notation of roots.

To make this appear, let the quantity $x^{\frac{1}{2}}$ ($= \sqrt{x}$) be propounded, in order to determine the fluxion thereof: which, by writing $\frac{1}{2}$ for p , in the general fluxion

fluxion $px^{p-1} \dot{x}$, comes out $\frac{1}{2} x^{\frac{1}{2}} \dot{x}$. Now, to prove that this is the true fluxion, put $y = x^{\frac{1}{2}}$, the quantity given; and then, by squaring both sides, you will have $y^2 = x$; which, in fluxions, gives $2y \dot{y} = \dot{x}$ (as has been already shewn.) This, by substituting for y , becomes $2x^{\frac{1}{2}} \dot{y} = \dot{x}$. Whence, by division, $\dot{y} = \frac{1}{2} x^{-\frac{1}{2}} \dot{x}$, the very same as before.

But, to demonstrate the same thing in a general manner, suppose the fluxion of x^n to be required (m and n being any whole numbers, whatever) put $y = x^{\frac{m}{n}}$; and then, by raising both sides to the power n , you will have $y^n = x^m$: which, in fluxions, gives $ny^{n-1} \dot{y} = mx^{m-1} \dot{x}$; and, consequently, $\dot{y} = \frac{m}{n} \times \frac{x^{m-1} \dot{x}}{y^{n-1}} = \frac{m}{n} \times$

$$\frac{yx^{m-1} \dot{x}}{y^n} = \frac{m}{n} \times \frac{yx^{m-1} \dot{x}}{x^m} = \frac{m}{n} \times \frac{x^{\frac{m}{n}} x^{m-1} \dot{x}}{x^m} = \frac{m}{n} \times x^{\frac{m}{n} - 1} \dot{x}.$$

Which value, of the fluxion of $x^{\frac{m}{n}}$, is evidently the very same, as that arising by expounding p in the general fluxion ($px^{p-1} \dot{x}$) by $\frac{m}{n}$; which was to be proved.

17. Now, from what has been thus far delivered, the following practical rules, for determining the fluxions of algebraic quantities, are obtained.

RULE I.

To find the Fluxion of any given Power of a flowing Quantity.

Multiply the fluxion of the root by the exponent of the given power, and the product into that power of the same

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root which arises by subtracting unity from the given exponent.

The reason of this rule is seen above; the rule itself being nothing more than p^{x-1} (the fluxion of x^p) expressed in words.

RULE II.

To find the Fluxion of the Product of several variable Quantities, multiplied together.

Multiply the fluxion of each, by the product of the rest of the quantities; so shall the sum of all the products thus arising be the true fluxion required.

The reason of which is, likewise, evident from what has been already delivered. See Art. 13.

RULE III.

To find the Fluxion of a Fraction, arising from the Division of one variable Quantity by another.

From the fluxion of the numerator, drawn into the denominator, subtract the fluxion of the denominator, drawn into the numerator; and divide the remainder by the square of the denominator.

This appears from $\frac{y^x - x^y}{yy}$, the fluxion of $\frac{x}{y}$, determined in Art. 12.

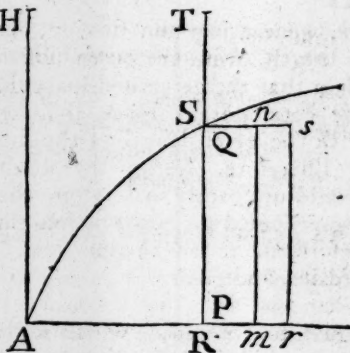
Though I might here, with propriety enough, put an end to this part, as my professed design therein extends no farther than giving the young beginner some account of the nature, and first principles, of fluxions, together with the investigation of the fundamental rules exhibited above; nevertheless, as different ways of bringing out the same truths have often a very good effect, and seeing fluxions of all quantities whatever (whether powers, fractions, &c.) are deducible from the fluxion of a rectangle, I shall, therefore, subjoin a different method, whereby the said fluxion

fluxion of a rectangle (*given by Prop. 2.*) may be investigated: in order to which it will be requisite, first of all, to premise the following

LEMMA.

18. *The Fluxion of a curvilineal Space ARS, generated by the Ordinate RS (or the intercepted Part of a Right-line RT (moving parallel to itself, is equal to the Rectangle (Rs) under the said Ordinate, and the Fluxion (Rr) of the Abscissa AR.*

For, let a right-line mn , of the same length with RS in the proposed position PQ, be conceived to move from thence, parallel to itself, with the same celerity that the generating line itself has in that position: by which means the rectangle $PrsQ$ will be uniformly generated, with the very celerity by which it begins to be generated, or, by which the space ARS is encreased in the proposed position PQ; since both the length, and velocity of mn , are the same as those of RS in the said position. Hence the rectangle so generated must, consequently, be the true fluxion of the space ARS, *by the definition.*



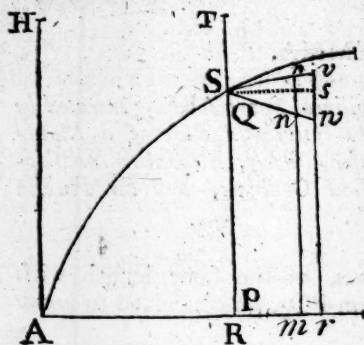
But the same thing may be otherwise made to appear, from a different, and more logical method of arguing; by proving that the required fluxion can neither be greater, nor less, than the said rectangle.

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Thus,

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Thus, if the line mn (while it moves uniformly on towards rs) be supposed to increase in length, the area $PmnQ$, generated thereby, will evidently be greater



than that which would uniformly arise in the same time, with the given length at the first position PQ ; since the new parts, produced each succeeding moment (as the generating line continues to lengthen) are greater and greater.

And, in the same manner, if the line mn , generating the fluxion, be supposed to decrease, in length, from the given position PQ , it is equally plain that the generated space $PmnQ$ will be less than the cotemporary space that would, uniformly, arise with the given length at the first position PQ .

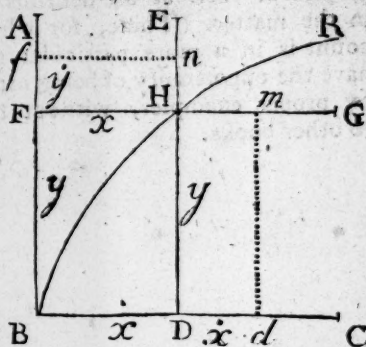
Therefore, seeing the fluxion (or the space that would uniformly arise from the generating celerity at the proposed position) is less than any space that can be described, in the given time, when the line mn increases, and greater than any space that can be described, when the said line decreases; it must consequently be equal to that space which will arise, when the length of the said line, from the given position, is supposed neither to increase nor decrease; that is, when the generated space $PmnQ$ is a rectangle, as in the preceding figure.

PROPOSITION

PROPOSITION III.

19. *To determine the Fluxion of the Product or Rectangle, of two variable Quantities (x and y.)*

Let two right-lines DE and FG, perpendicular to each other, be conceived to move from two other right-lines BA and BC, continually parallel to themselves, and thereby generate the variable rectangle DF: let the path of their intersection, or the place of the angle H, be the line BHR, dividing the generated rectangle DF in two parts, BDH and BHF: moreover let Dd ($\dot{x}$) and Ff ($\dot{y}$) be the fluxions of the sides BD (x) and BF (y); and suppose dm and fn to be drawn parallel, and equal, to DH and FH, respectively.



Then, since, by the preceding lemma, the fluxion of the space or area BDH, is truly expressed by the rectangle Dm ($=x\dot{y}$), and that of the space or area BHF, by the rectangle Fn ($=y\dot{x}$), it follows, because equal quantities have equal fluxions, that the fluxion of the proposed rectangle xy ($=BDH+BHF$) is truly expressed by $x\dot{y}+y\dot{x}$, the very expression before determined.
See Art. II.

It may, perhaps, be expected, that I should now give some instances of the use and application of the theory hitherto explained, in the resolution of problems: but, having insisted very largely on this head in my *Doctrine and Application of Fluxions*, I shall take the liberty to recommend that work, to the perusal of such as are desirous of farther information in the matter. Those, for whose use the above account is in a more particular manner designed, may have the opportunity of being instructed in the practice, by proper examples, without the trouble of turning to other books:

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